SPECTRAL PROPERTIES OF BOUNDARY-VALUE-TRANSMISSION PROBLEMS WITH A CONSTANT RETARDED ARGUMENT

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ABSTRACT. In this work, spectrum and asymptotics of eigenfunctions of a generalized class of boundary value problems with constant retarded argument are obtained. Contrary to previous works in the literature, the problem has non-classical transmission conditions.
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KEYWORDS AND PHRASES. Differential equation with retarded argument; non-classical transmission conditions; asymptotics of eigenvalues and eigenfunctions.

1. FORMULATION OF THE PROBLEM

Delay differential equations provide a mathematical model for physical, biological systems in which the rate of change of the system depends upon their past history. Differential equations with retarded argument is an active research area of delay differential equations. Each year there is an increase in the number of articles devoted to the study of various applied problems formulated with the use of delays. However, in an overwhelming majority of applied articles, constant delays are considered. Such a consideration is an improvement compared with the model of an "ideal" process which is obtained if it is assumed that there are no delays at all, that the "functioning" takes place instantly [1].

In this study we shall investigate the eigenvalue problem $L := L\left(q; a, \lambda, r; h, H, d_j\right)$ $(j = 1, 2, 3)$ which consists of Sturm-Liouville equation

$$-y''(x) + q(x)y(x-a) = \lambda^2 r(x)y(x) \tag{1}$$

on $\Lambda = \cup\Lambda^{\pm}$ with boundary conditions

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y'(T) + Hy(T) = 0 \tag{3}$$

and non-classical transmission conditions

$$y(c+0) = d_1 y(c-0), \tag{4}$$

$$y'(c+0) = d_2 y'(c-0) + d_3 y(c-0) \tag{5}$$

where $r(x) = \frac{1}{r_1^2}$ for $x \in \Lambda^- = [0, c)$ and $r(x) = \frac{1}{r_2^2}$ for $x \in \Lambda^+ = (c, T]$; the real-valued function $q(x)$ is continuous in Λ and has a finite limit $q(c \pm 0) =$

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$\lim_{x\to c\pm 0} q(x)$, $x-a\geq 0$, if $x\in\Lambda^-; x-a\geq c$, if $x\in\Lambda^+$; λ is a real spectral parameter; a, r_i $(i=1,2)$, h, H, d_j $(j=1,2,3)$ are arbitrary real numbers such that $r_1r_2d_1d_2\neq 0$ and $d_1r_2=d_2r_1$.

The main goal of this paper is to study the spectrum and asymptotics of eigenfunctions of the problem L. Spectral properties of differential equations with retarded argument which contain such a generalized transmission conditions $(d_3\neq 0)$ have not been studied yet. So, the results obtained in this work are extension and generalization of previous works in the literature. For example, if we take $a=0$ and/or $d_3=0$ and/or $r(x)\equiv 1$ then the asymptotic formulas for eigenvalues and eigenfunctions correspond to those for the classical Sturm-Liouville problem [1-19]. Moreover, results and methods of these kind of problems can be useful for investigating the inverse problems for ordinary and partial differential equations (see, e.g., [19]).

We want to also note that for linear equations with constant delay, very effective operational methods are available (e.g., first of all the Laplace transform) [1].

Let $\vartheta^-(x,\lambda)$ be a solution of Eq. (1) on $\overline{\Lambda^-}=\Lambda^-\cup\{c\}$, satisfying the initial conditions

$$\vartheta^-(0,\lambda)=1,\ \frac{\partial\vartheta^-(0,\lambda)}{\partial x}=h. \tag{6}$$

The conditions (6) define a unique solution of Eq. (1) on $\overline{\Lambda^-}$.

After defining the above solution we shall define the solution $\vartheta^+(x,\lambda)$ of Eq. (1) on $\overline{\Lambda^+}=\Lambda^+\cup\{c\}$ by means of the solution $\vartheta^-(x,\lambda)$ using the initial conditions

$$\vartheta^+(c+,\lambda)=d_1\vartheta^-(c-,\lambda), \tag{7}$$

$$\frac{\partial\vartheta^+(c+,\lambda)}{\partial x}=\left[d_2\frac{\partial\vartheta^-(c-,\lambda)}{\partial x}+d_3\vartheta^-(c-,\lambda)\right] \tag{8}$$

The conditions (7)-(8) are defined as a unique solution of Eq. (1) on $\overline{\Lambda^+}$.

Consequently, the function $\vartheta(x,\lambda)$ is defined on Λ by the equality

$$\vartheta(x,\lambda)=\begin{cases}\vartheta^-(x,\lambda), & x\in\Lambda^-,\\ \vartheta^+(x,\lambda), & x\in\Lambda^+\end{cases}$$

is a solution of the Eq. (1) on Λ; which satisfies one of the boundary conditions and both transmission conditions.

2. Asymptotics of Eigenvalues and Eigenfunctions of the problem L

We begin by writing the problem L in terms of the following equivalent integral equations.

Lemma 1. *Let $\vartheta(x,\lambda)$ be a solution of Eq.(1) and $\lambda>0$. Then the following integral equations hold:*

$$\vartheta^-(x,\lambda)=\cos\lambda r_1x+\frac{h}{r_1\lambda}\sin\lambda r_1x+\frac{r_1}{\lambda}\int_0^x q(\tau)\sin\lambda r_1(x-\tau)\vartheta^-(\tau-a,\lambda)\,d\tau, \tag{9}$$

$$\vartheta^+(x,\lambda)=d_1\vartheta^-(c-,\lambda)\cos\lambda r_2(x-c)+\frac{1}{\lambda r_2}\left[d_2\frac{\partial\vartheta^-(c-,\lambda)}{\partial x}+d_3\vartheta^-(c-,\lambda)\right]$$

(10) $$\times \sin \lambda r_2 (x-c) + \frac{r_2}{\lambda} \int\limits_{c+}^{x} q(\tau) \sin \lambda r_2 (x-\tau)\, \vartheta^{+}(\tau - a, \lambda)\, d\tau.$$

Proof. To prove this, it is enough to substitute $\lambda^2 \vartheta^{\pm}(\tau,\lambda) + \frac{\partial^2 \vartheta^{\pm}(\tau,\lambda)}{\partial \tau^2}$ instead of $q(\tau)\vartheta^{\pm}(\tau - a, \lambda)$ in (9) and (10) respectively and integrate by parts twice. □

Theorem 1. *The problem L can have only simple eigenvalues.*

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of the problem L and

$$\widetilde{y}(x,\widetilde{\lambda}) = \begin{cases} \widetilde{y}_1(x,\widetilde{\lambda}), & x \in \Lambda^-, \\ \widetilde{y}_2(x,\widetilde{\lambda}), & x \in \Lambda^+ \end{cases}$$

be a corresponding eigenfunction. Then from (2) and (6) it follows that

$$W\left[\widetilde{y}_1(0,\widetilde{\lambda}), \vartheta^{-}(0,\widetilde{\lambda})\right] = \begin{vmatrix} \widetilde{y}_1(0,\widetilde{\lambda}) & -h \\ \widetilde{y}_1'(0,\widetilde{\lambda}) & 1 \end{vmatrix} = 0,$$

and the functions $\widetilde{y}_1(x,\widetilde{\lambda})$ and $\vartheta^{-}(x,\widetilde{\lambda})$ are linearly dependent on $\overline{\Lambda^-}$. Here $W[f,g]$ denotes the Wronskians of the functions f and g. We can also prove that the functions $\widetilde{y}_2(x,\widetilde{\lambda})$ and $\vartheta^{+}(x,\widetilde{\lambda})$ are linearly dependent on $\overline{\Lambda^+}$. Hence

(11) $$\widetilde{y}_j(x,\widetilde{\lambda}) = R_j w_j(x,\widetilde{\lambda}) \qquad (j=1,2)$$

for some $R_1 \neq 0$ and $R_2 \neq 0$. We must show that $R_1 = R_2$. Suppose that $R_1 \neq R_2$. From the equalities (4) and (11), we have

$$\begin{aligned} \widetilde{y}(c+0,\widetilde{\lambda}) - d_1 \widetilde{y}(c-0,\widetilde{\lambda}) &= \widetilde{y}_2(c+0,\widetilde{\lambda}) - d_1 \widetilde{y}_1(c-0,\widetilde{\lambda}) \\ &= R_2 \vartheta^{+}(c,\widetilde{\lambda}) - R_1 \vartheta^{+}(c,\widetilde{\lambda}) \\ &= (R_1 - R_2)\, \vartheta^{+}(c,\widetilde{\lambda}) = 0. \end{aligned}$$

Since $(R_1 - R_2) \neq 0$ it follows that

(12) $$\vartheta^{+}\left(c,\widetilde{\lambda}\right) = 0$$

and since $\vartheta^{+}\left(c,\widetilde{\lambda}\right) = d_1 \vartheta^{-}\left(c,\widetilde{\lambda}\right)$ and $\vartheta^{-}\left(c,\widetilde{\lambda}\right) = \widetilde{y}(c-0,\widetilde{\lambda})$ we have

(13) $$\widetilde{y}(c-0,\widetilde{\lambda}) = 0.$$

By the same procedure, from equality (5) and (13) we can derive that

$$\begin{aligned} &\widetilde{y}'(c+0,\widetilde{\lambda}) - d_2 \widetilde{y}'(c-0,\widetilde{\lambda}) - d_3 \widetilde{y}(c-0,\widetilde{\lambda}) \\ &= R_2 \frac{\partial \vartheta^{+}(c,\widetilde{\lambda})}{\partial x} - R_1 d_2 \frac{\partial \vartheta^{-}(c,\widetilde{\lambda})}{\partial x} \\ &= R_2 \frac{\partial \vartheta^{+}(c,\widetilde{\lambda})}{\partial x} - R_1 \frac{\partial \vartheta^{+}(c,\widetilde{\lambda})}{\partial x} \\ &= (R_1 - R_2)\, \frac{\partial \vartheta^{+}(c,\widetilde{\lambda})}{\partial x} = 0. \end{aligned}$$

Since $(R_1 - R_2) \neq 0$ it follows that

(14) $$\frac{\partial \vartheta^{+}(c,\widetilde{\lambda})}{\partial x} = 0.$$

From the fact that $\vartheta^{+}(x,\widetilde{\lambda})$ is a solution of the Eq. (1) on $\overline{\Lambda^{+}}$ and satisfies the initial conditions (12) and (14) it follows that $\vartheta^{-}(x,\widetilde{\lambda})=0$ identically on $\overline{\Lambda^{+}}$ (see [1, p. 12, Theorem 1.2.1]).

By using this method we may also find

$$\vartheta^{-}\left(c,\widetilde{\lambda}\right)=\frac{\partial\vartheta^{-}(c,\widetilde{\lambda})}{\partial x}=0.$$

From the latter discussions of $\vartheta^{+}(x,\widetilde{\lambda})$ it follows that $\vartheta^{-}(x,\widetilde{\lambda})=0$ identically on Λ. But this contradicts (6), thus completing the proof. □

Let $q_1=\int\limits_{\overline{\Lambda^{-}}} q(\tau)\,d\tau$ and $q_2=\int\limits_{\overline{\Lambda^{+}}} q(\tau)\,d\tau$.

Lemma 2. (1) *Let $\lambda\geq 2q_1$. Then for the solution $\vartheta^{-}(x,\lambda)$ of Eq. (8), the following inequality holds*:

$$\left|\vartheta^{-}(x,\lambda)\right|\leq\frac{2}{2-r_1}\sqrt{1+\frac{h^2}{4r_1^2q_1^2}},\quad x\in\overline{\Lambda^{-}}. \tag{15}$$

(2) *Let $\lambda\geq\max\{2q_1,2q_2\}$. Then for the solution $\vartheta^{+}(x,\lambda)$ of Eq. (9), the following inequality holds*:

$$\left|\vartheta^{+}(x,\lambda)\right|\leq\frac{\sqrt{4q_1^2r_1^2+h^2}}{(2-r_2)\,q_1}\left[\frac{2d_1}{(2-r_1)\,r_1}+\frac{d_2}{r_2}\left(1+\frac{2r_1^2q_1^2}{2-r_1}\right)+\frac{d_3}{2\,(2-r_1)\,r_1r_2q_1}\right],\quad x\in\overline{\Lambda^{+}}. \tag{16}$$

Proof. Let $B_{1\lambda}=\sup_{\overline{\Lambda^{-}}}\left|\vartheta^{-}(x,\lambda)\right|$. Then from (9), it follows that, for every $\lambda>0$, the following inequality holds:

$$B_{1\lambda}\leq\sqrt{1+\frac{h^2}{r_1^2\lambda^2}}+\frac{r_1}{\lambda}q_1B_{1\lambda}.$$

Thus, if $\lambda\geq 2q_1$ we get (15).

Differentiating (9) with respect to x, we have

$$\frac{\partial\vartheta^{-}(x,\lambda)}{\partial x}=-\lambda r_1\sin\lambda r_1x+h\cos\lambda r_1x+r_1^2\int\limits_0^x q(\tau)\cos\lambda r_1\,(x-\tau)\,\vartheta^{-}(\tau-a,\lambda)d\tau. \tag{17}$$

From (15) and (17), it follows that, for $\lambda\geq 2q_1$, the following inequality holds:

$$\left|\lambda^{-1}\frac{\partial\vartheta^{-}(x,\lambda)}{\partial x}\right|\leq\sqrt{r_1^2+\frac{h^2}{4q_1^2}}\left(1+\frac{2r_1^2q_1^2}{2-r_1}\right). \tag{18}$$

Let $B_{2\lambda}=\sup_{\overline{\Lambda^{+}}}\left|\vartheta^{+}(x,\lambda)\right|$. Then from (10), (15) and (18) it follows that, for $\lambda\geq 2q_1$ and $\lambda\geq 2q_2$, the following inequalities holds:

$$\begin{aligned}B_{2\lambda}\ \leq\ & d_1\frac{2}{2-r_1}\sqrt{1+\frac{h^2}{4r_1^2q_1^2}}+\frac{d_2}{r_2}\sqrt{r_1^2+\frac{h^2}{4q_1^2}}\left(1+\frac{2r_1^2q_1^2}{2-r_1}\right)\\&+\frac{d_3}{\lambda r_2}\frac{2}{2-r_1}\sqrt{1+\frac{h^2}{4r_1^2q_1^2}}B_{2\lambda}.\end{aligned}$$

Hence if $\lambda\geq\max\{2q_1,2q_2\}$ we get (16). □

From Lemma 1, using the well-known successive approximation method, it is easy to obtain the following asymptotic expressions of fundamental solutions.

Lemma 3. *The following asymptotic estimates*

$$\vartheta^{-}(x,\lambda) = \cos\lambda r_1 x + O\left(\frac{1}{\lambda}\right), \tag{19}$$

$$\frac{\partial\vartheta^{-}(x,\lambda)}{\partial x} = -\lambda r_1 \sin\lambda r_1 x + O(1), \tag{20}$$

$$\vartheta^{+}(x,\lambda) = d_1 \cos\lambda\left(r_2 x + c\left(r_1 - r_2\right)\right) + O\left(\frac{1}{\lambda}\right), \tag{21}$$

$$\frac{\partial\vartheta^{+}(x,\lambda)}{\partial x} = -d_1 r_2 \lambda \sin\lambda\left(r_2 x + c\left(r_1 - r_2\right)\right) + O(1) \tag{22}$$

are valid as $\lambda \to \infty$.

The function $\vartheta(x,\lambda)$ defined in introduction is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting $\vartheta(x,\lambda)$ into (3), we get the characteristic equation

$$\Xi(\lambda) \equiv \frac{\partial\vartheta^{+}(T,\lambda)}{\partial x} + H\vartheta^{+}(T,\lambda) = 0. \tag{23}$$

Thus the set of eigenvalues of boundary-value problem L coincides with the set of real roots of Eq. (23). From now on, without loss of generality we shall consider only the case $hH \neq 0$. The other cases may be considered analogically.

Theorem 2. *The problem L has an infinite set of positive eigenvalues.*

Proof. Putting the expressions (19)-(22) into (23), we get

$$\Xi(\lambda) \equiv -d_1 r_2 \lambda \sin\lambda\left(r_2 T + c\left(r_1 - r_2\right)\right) + O(1)$$

$$+H\left(d_1 \cos\lambda\left(r_2 T + c\left(r_1 - r_2\right)\right) + O\left(\frac{1}{\lambda}\right)\right)$$

$$= -d_1 r_2 \lambda \sin\lambda\left(r_2 T + c\left(r_1 - r_2\right)\right) + O(1) = 0. \tag{24}$$

Let λ be sufficiently large. Obviously, for large λ, Eq. (24) has, evidently, an infinite set of roots. The proof is complete. □

By Theorem 2 we conclude that the problem L has infinitely many nontrivial solutions. Let n be a natural number. We shall say that the number λ is situated near the number $\frac{n\pi}{(r_2 T + c(r_1 - r_2))}$ if $\left|\frac{n\pi}{(r_2 T + c(r_1 - r_2))} - \lambda\right| < \frac{1}{2}$.

Theorem 3. *Let n be a natural number. For each sufficiently large n, there is exactly one eigenvalue of the problem L near* $\frac{n\pi}{(r_2 T + c(r_1 - r_2))}$.

Proof. We consider the expression which is denoted by $O(1)$ in the Eq. (24):

$$\begin{aligned}
&((d_1+d_2)\cos\lambda r_2(T-c)+d_3\sin\lambda r_2(T-c)) \\
&\times\left(h\cos\lambda r_1 c+r_1^2\int_0^c q(\tau)\cos\lambda r_1(c-\tau)\,\vartheta^-(\tau-a,\lambda)\,d\tau\right) \\
&+(-d_1 r_2\lambda\sin\lambda r_2(T-c)+d_3 r_2\lambda\cos\lambda r_2(T-c)) \\
&\times\left(\frac{h}{r_1\lambda}\sin\lambda r_1 c+\frac{r_1}{\lambda}\int_0^c q(\tau)\sin\lambda r_1(c-\tau)\,\vartheta^-(\tau-a,\lambda)\,d\tau\right) \\
&+\frac{d_2}{r_2}\sin\lambda r_2(T-c)\left(-r_1 h\sin\lambda r_1 c-r_1^3\int_0^c q(\tau)\sin\lambda r_1(c-\tau)\,\vartheta^-(\tau-a,\lambda)\,d\tau\right) \\
&+r_2^2\int_{c^+}^T q(\tau)\cos\lambda r_2(T-\tau)\,\vartheta^+(\tau-a,\lambda)\,d\tau.
\end{aligned}$$

If inequalities (15)-(16) are taken into consideration, it can be shown by differentiation with respect to λ that for large λ this expression has bounded derivative.It is obvious that for large λ the roots of Eq. (24) are situated close to entire numbers. We shall show that, for large n, only one root (24) lies near to each $\frac{n\pi}{(r_2T+c(r_1-r_2))}$. We consider the function $J(\lambda)=\lambda\sin\lambda(r_2T+c(r_1-r_2))+O(1)$. Its derivative, which has the form

$$J'(\lambda)=\sin\lambda(r_2T+c(r_1-r_2))+\lambda(r_2T+c(r_1-r_2))\cos\lambda(r_2T+c(r_1-r_2))+O(1),$$

does not vanish for λ close to $\frac{n\pi}{(r_2T+c(r_1-r_2))}$ for sufficiently large n. Thus our assertion follows by Rolle's Theorem. □

Let n be sufficiently large. the eigenvalue of the problem L situated near $\frac{n\pi}{(r_2T+c(r_1-r_2))}$. We set $\lambda_n=\frac{n\pi}{(r_2T+c(r_1-r_2))}+\delta_n$. Then from (24) it follows that $\delta_n=O\left(\frac{1}{n}\right)$. Consequently

$$\lambda_n=\frac{n\pi}{(r_2T+c(r_1-r_2))}+O\left(\frac{1}{n}\right). \tag{25}$$

The formula (25) make it possible to obtain asymptotic expressions for eigenfunction of the problem L. Now we are ready to present asymptotic expressions of eigenfunctions. By putting (25) in the (19) and (21), we have the next Theorem.

Theorem 4. *The following asymptotic formulas hold for eigenfunctions of boundary-value-transmission problem L for each $x\in\Lambda$:*

$$\vartheta^-(x,\lambda_n)=\cos\frac{n\pi r_1 x}{r_2T+c(r_1-r_2)}+O(\frac{1}{n}),$$
$$\vartheta^+(x,\lambda_n)=d_1\cos\frac{n\pi(r_2x+c(r_1-r_2))}{r_2T+c(r_1-r_2)}+O(\frac{1}{n}).$$

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