Inclusion properties of Lucas polynomials for bi-univalent functions introduced through the $\mathfrak{q}$-analogue of Noor integral operator

Şahsene Altınkaya
Department of Mathematics, Faculty of Arts and Science,
Bursa Uludag University, 16059-Bursa, Turkey.
sahsenealtinkaya@gmail.com

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Abstract

In this current paper, by using the $(\mathbf{P},\mathbf{Q})$-Lucas polynomials and the $\mathfrak{q}$-analogue of Noor integral operator, we aim to build a bridge between the Theory of Geometric Functions and that of Special Functions.

Keywords: $(\mathbf{P},\mathbf{Q})$-Lucas polynomials, coefficient bounds, bi-univalent functions.

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1 Introduction, preliminaries, known results

In modern science there is a huge interest in the theory and application of the Fibonacci polynomials, the Lucas polynomials, the Chebychev polynomials, the Tschebyscheff polynomials, the Pell polynomials, the Pell-Lucas polynomials, the Lucas-Lehmer polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations. These polynomials play a major role in a variety of disciplines in the mathematical, physical, statistical and engineering sciences (see, for example, [4, 8, 11, 14, 18, 19]) and references therein.

Definition 1 *(See [9]) Let $\mathbf{P}(\varkappa)$ and $\mathbf{Q}(\varkappa)$ represent polynomials with real coefficients. The $(\mathbf{P},\mathbf{Q})$-Lucas polynomials $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$ are introduced by the recurrence relevance*

$$L_{\mathbf{P},\mathbf{Q},j}(\varkappa)=\mathbf{P}(\varkappa)L_{\mathbf{P},\mathbf{Q},j-1}(\varkappa)+\mathbf{Q}(\varkappa)L_{\mathbf{P},\mathbf{Q},j-2}(\varkappa)\quad (j\geq 2),$$

with the first few terms of the $(\mathbf{P},\mathbf{Q})$*-Lucas polynomials are as follows*

$$\begin{aligned}
&L_{\mathbf{P},\mathbf{Q},0}(\varkappa)=2,\\
&L_{\mathbf{P},\mathbf{Q},1}(\varkappa)=\mathbf{P}(\varkappa),\\
&L_{\mathbf{P},\mathbf{Q},2}(\varkappa)=\mathbf{P}^2(\varkappa)+2\mathbf{Q}(\varkappa),\\
&L_{\mathbf{P},\mathbf{Q},3}(\varkappa)=\mathbf{P}^3(\varkappa)+3\mathbf{P}(\varkappa)\mathbf{Q}(\varkappa),\\
&\vdots
\end{aligned} \tag{1}$$

Table 1: Specific cases of the $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$

$\mathbf{P}(\varkappa)$	$\mathbf{Q}(\varkappa)$	$L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$
$\varkappa$	1	Lucas polynomials $L_n(\varkappa)$
$2\varkappa$	1	Pell-Lucas polynomials $D_n(\varkappa)$
1	$2\varkappa$	Jacobsthal-Lucas polynomials $j_n(\varkappa)$
$3\varkappa$	-2	Fermat-Lucas polynomials $f_n(\varkappa)$
$2\varkappa$	-1	Chebyshev polynomials $T_n(\varkappa)$

Theorem 2 *(See [9]) Let* $\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)$ *indicate the generating function of the* $(\mathbf{P},\mathbf{Q})$*-Lucas polynomial sequence* $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$*. Then*

$$\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)=\sum_{j\geq 0}^{\infty}L_{\mathbf{P},\mathbf{Q},j}(\varkappa)z^j=\frac{2-\mathbf{P}(\varkappa)z}{1-\mathbf{P}(\varkappa)z-\mathbf{Q}(\varkappa)z^2}.$$

Let $\mathcal{A}$ indicate the class of functions $\mathfrak{f}$ normalized by

$$\mathfrak{f}(z)=z+a_2z^2+a_3z^3+\cdots, \tag{2}$$

which are analytic in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$. The class of this kind of functions is represented by S.

With a view to reminding the rule of subordination for analytic functions, let $\mathfrak{f}$, $\mathfrak{g}$ be analytic in Δ. A function $\mathfrak{f}$ is *subordinate* to $\mathfrak{g}$, indited as $\mathfrak{f}\prec\mathfrak{g}$, if there exists a function τ, analytic in Δ, such that

$$\tau(0)=0,\ \ |\tau(z)|<1$$

and

$$\mathfrak{f}(z)=\mathfrak{g}(\tau(z))\qquad(z\in\Delta).$$

According to the *Koebe-One Quarter Theorem*, every univalent function $\mathfrak{f}\in\mathcal{A}$ has an inverse $\mathfrak{f}^{-1}$ satisfying

$$\mathfrak{f}^{-1}(\mathfrak{f}(z))=z$$

and

$$\mathfrak{f}\left(\mathfrak{f}^{-1}(w)\right)=w\ \ \left(|w|<\frac{1}{4}\right)$$

where

$$\mathfrak{g}(\mathfrak{w}) = \mathfrak{f}^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (3)$$

A function $\mathfrak{f} \in \mathcal{A}$ is called *bi-univalent* in Δ if both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are univalent in Δ. Let $\mathbf{\Sigma}$ denote the class of bi-univalent functions in Δ given by (2). For several interesting examples in the class $\mathbf{\Sigma}$, see [16] (see also [1, 2, 3, 6, 10, 12, 15]).

It may be of interest to recall that Srivastava used the basic (or $\mathfrak{q}$-) hypergeometric functions in a book chapter (see, for details, [17]). Thus, the theory of univalent functions was characterized by concept of the $\mathfrak{q}$-calculus. For the simplicity, we ensure some fundamental description of $\mathfrak{q}$-calculus which are used in this paper. Next, we remind some identities of fractional $\mathfrak{q}$-calculus operators of complex valued function $\mathfrak{f}$.

For $0 < \mathfrak{q} < 1$, the $\mathfrak{q}$-derivative of a function $\mathfrak{f} \in \mathcal{A}$ is defined as follows

$$\partial_{\mathfrak{q}} \mathfrak{f}(z) = \frac{\mathfrak{f}(z) - \mathfrak{f}(\mathfrak{q}z)}{z(1-\mathfrak{q})}, \qquad (z \in \Delta).$$

Obviously, for $j \in \mathbb{N} := \{1, 2, \ldots\}$ and $z \in \Delta$

$$\partial_{\mathfrak{q}} \left(\sum_{j \geq 1}^{\infty} a_j z^j \right) = \sum_{j \geq 1}^{\infty} [j, \mathfrak{q}] \, a_j z^{j-1},$$

where

$$[j, \mathfrak{q}] = \frac{1 - \mathfrak{q}^j}{1 - \mathfrak{q}}, \qquad [0, \mathfrak{q}] = 0.$$

Moreover, it is worth mentioning that

$$[j, \mathfrak{q}]! = \begin{cases} 1, & j = 0 \\ [1, \mathfrak{q}] \, [2, \mathfrak{q}] \, [3, \mathfrak{q}] \ldots [j, \mathfrak{q}], & j \in \mathbb{N} \end{cases} .$$

Also the $\mathfrak{q}$-generalized Pochhammer symbol for $\wp > 0$ is given by

$$[\wp, \mathfrak{q}]_j = \begin{cases} 1, & j = 0 \\ [\wp, \mathfrak{q}] \, [\wp + 1, \mathfrak{q}] \ldots [\wp + j - 1, \mathfrak{q}], & j \in \mathbb{N} \end{cases} .$$

Very recently, Arif et al. [5] defined the function $\digamma_{\mathfrak{q},\mu+1}^{-1}(z)$ by

$$\digamma_{\mathfrak{q},\mu+1}^{-1}(z) * \digamma_{\mathfrak{q},\mu+1}(z) = z \partial_{\mathfrak{q}} \mathfrak{f}(z) \quad (\mu > -1),$$

where the function $\digamma_{\mathfrak{q},\mu+1}(z)$ is given by

$$\digamma_{\mathfrak{q},\mu+1}(z) = z + \sum_{j=2}^{\infty} \frac{[\mu + 1, \mathfrak{q}]_{j-1}}{[j-1, \mathfrak{q}]!} z^j, \qquad (z \in \Delta). \quad (4)$$

Because of the series defined in (4) is convergent absolutely in Δ, by making use of the description of $\mathfrak{q}$-derivative through convolution, we now define the integral operator $\zeta_{\mathfrak{q}}^{\mu}:\Delta\rightarrow\Delta$ by

$$\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)=F_{\mathfrak{q},\mu+1}^{-1}(z)*\mathfrak{f}(z)=z+\sum_{j=2}^{\infty}\Theta_{j-1}a_{j}z^{j},\qquad(z\in\Delta),\tag{5}$$

where

$$\Theta_{j-1}=\frac{[j,\mathfrak{q}]!}{[\mu+1,\mathfrak{q}]_{j-1}}.$$

We note that $\zeta_{\mathfrak{q}}^{0}\mathfrak{f}(z)=z\partial_{\mathfrak{q}}\mathfrak{f}(z),\ \zeta_{\mathfrak{q}}^{1}\mathfrak{f}(z)=\mathfrak{f}(z)$, and

$$\lim_{\mathfrak{q}\rightarrow1^{-}}\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)=z+\sum_{j=2}^{\infty}\frac{j!}{(\mu+1)_{j-1}}a_{j}z^{j}.$$

This shows that, by taking $\mathfrak{q}\rightarrow1^{-}$, the operator defined in (5) reduces to the familiar Noor integral operator introduced in [13].

We want to assert evidently that, in the paper, by using the $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$ functions, our methodology constructions a bridge between the Theory of Geometric Functions and that of Special Functions. Thus, we aim to introduce a new class of bi-univalent functions introduced through the $(\mathbf{P},\mathbf{Q})$-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szegö problem for this new function class.

Definition 3 *A function $\mathfrak{f}\in\mathbf{\Sigma}$ is said to be in the class*

$$U_{\mathbf{\Sigma}}^{\mu}(\mathfrak{q};\varkappa)\quad(\mu>-1,\ 0<\mathfrak{q}<1,\ z,w\in\Delta)$$

if the following subordinations are fulfilled:

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)\right)}{\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)}\prec\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)-1$$

and

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta_{\mathfrak{q}}^{\mu}\mathfrak{g}(w)\right)}{\zeta_{\mathfrak{q}}^{\mu}\mathfrak{g}(w)}\prec\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w)-1$$

where the function $\mathfrak{g}$ is given by (3).

Remark 4 *Upon setting $\mathfrak{q}\rightarrow1^{-}$in Definition 3, it is readily seen that a function $\mathfrak{f}\in\mathbf{\Sigma}$ is in the class*

$$U_{\mathbf{\Sigma}}^{\mu}(\varkappa)\quad(\mu>-1,\ z,w\in\Delta)$$

if the following conditions are fulfilled:

$$\frac{z\left(\zeta^{\mu}\mathfrak{f}(z)\right)'}{\zeta^{\mu}\mathfrak{f}(z)}\prec\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)-1$$

$$\frac{z\left(\zeta^{\mu}\mathfrak{g}(w)\right)'}{\zeta^{\mu}\mathfrak{g}(w)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w)-1$$

where the function $\mathfrak{g}$ *is given by (3).*

Remark 5 *Upon setting* $\mu=1$ *in Definition 3, it is readily seen that a function* $\mathfrak{f}\in\mathbf{\Sigma}$ *is in the class*

$$S_{\mathbf{\Sigma}}(\varkappa) \quad (z,w\in\Delta)$$

if the following conditions are fulfilled:

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)-1$$

and

$$\frac{w\mathfrak{g}'(w)}{\mathfrak{g}(w)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w)-1$$

where the function $\mathfrak{g}$ *is given by (3).*

Remark 6 *Upon setting* $\mu=0$ *in Definition 3, it is readily seen that a function* $\mathfrak{f}\in\mathbf{\Sigma}$ *is in the class*

$$C_{\Sigma}(\varkappa) \quad (z,w\in\Delta)$$

if the following conditions are fulfilled:

$$\left(1+\frac{z\mathfrak{f}''(z)}{\mathfrak{f}(z)}\right) \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)-1$$

and

$$\left(1+\frac{w\mathfrak{g}''(w)}{\mathfrak{g}(w)}\right) \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w)-1$$

where the function $\mathfrak{g}$ *is given by (3).*

2 Inequalities for the Taylor-Maclaurin Coefficients

Theorem 7 *Let the function* $\mathfrak{f}$ *given by (2) be in the class* $U^{\mu}_{\mathbf{\Sigma}}(\mathfrak{q};\varkappa)$*. Then*

$$|a_2| \leq \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{|(\mathfrak{q}+1)(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-2\mathfrak{q}\Theta_1^2\mathbf{Q}(\varkappa)|\,\mathfrak{q}}}$$

and

$$|a_3| \leq \frac{\mathbf{P}^2(\varkappa)}{\mathfrak{q}^2\Theta_1^2}+\frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}.$$

Proof. Let $\mathfrak{f} \in U^{\mu}_{\mathbf{\Sigma}}(\mathfrak{q};\varkappa)$. From Definition 3, for some analytic functions Φ, Ψ such that

$$\Phi(0) = 0,\ |\Phi(z)| = \left|m_1 z + m_2 z^2 + m_3 z^3 + \cdots\right| < 1 \quad (z \in \Delta),$$

$$\Psi(0) = 0,\ |\Psi(w)| = \left|n_1 w + n_2 w^2 + n_3 w^3 + \cdots\right| < 1 \quad (w \in \Delta),$$

we can write

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)\right)}{\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)} = \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(\Phi(z)) - 1$$

and

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta^{\mu}_{\mathfrak{q}}\mathfrak{g}(w)\right)}{\zeta^{\mu}_{\mathfrak{q}}\mathfrak{g}(w)} = \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(\Psi(w)) - 1,$$

or equivalently

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)\right)}{\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)} = -1 + L_{\mathbf{P},\mathbf{Q},0}(\varkappa) + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\Phi(z) + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\Phi^2(z) + \cdots \tag{6}$$

and

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta^{\mu}_{\mathfrak{q}}\mathfrak{g}(w)\right)}{\zeta^{\mu}_{\mathfrak{q}}\mathfrak{g}(w)} = -1 + L_{\mathbf{P},\mathbf{Q},0}(\varkappa) + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\Psi(w) + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\Psi^2(w) + \cdots. \tag{7}$$

From the equalities (6) and (7), we obtain that

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)\right)}{\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)} = 1 + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_1 z + \left[L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)m_1^2\right] z^2 + \cdots \tag{8}$$

and

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta^{\mu}_{\mathfrak{q}}\mathfrak{g}(w)\right)}{\zeta^{\mu}_{\mathfrak{q}}\mathfrak{g}(w)} = 1 + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_1 w + \left[L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)n_1^2\right] w^2 + \cdots. \tag{9}$$

Additionally, it is well known that

$$|m_k| \le 1 \quad \text{and} \quad |n_k| \le 1 \quad (k \in \mathbb{N}).$$

By comparing the corresponding coefficients in (8) and (9), we have

$$\mathfrak{q}\Theta_1 a_2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_1, \tag{10}$$

$$\mathfrak{q}(\mathfrak{q}+1)\Theta_2 a_3 - \mathfrak{q}\Theta_1^2 a_2^2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)m_1^2, \tag{11}$$

$$-\mathfrak{q}\Theta_1 a_2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_1 \tag{12}$$

and

$$\mathfrak{q}(\mathfrak{q}+1)\Theta_2(2a_2^2 - a_3) - \mathfrak{q}\Theta_1^2 a_2^2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)n_1^2. \tag{13}$$

From the equations (10) and (12), we can easily see that

$$m_1 = -n_1, \tag{14}$$

$$2\mathfrak{q}^2\Theta_1^2 a_2^2 = L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left(m_1^2+n_1^2\right). \tag{15}$$

If we add (11) to (13), we get

$$2\mathfrak{q}\left[(\mathfrak{q}+1)\Theta_2-\Theta_1^2\right]a_2^2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left(m_2+n_2\right)+L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\left(m_1^2+n_1^2\right). \tag{16}$$

By using (15) in the equality (16), we have

$$2\mathfrak{q}\left\{\left[(\mathfrak{q}+1)\Theta_2-\Theta_1^2\right]L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa)-\mathfrak{q}\Theta_1^2 L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\right\}a_2^2 = L^3_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left(m_2+n_2\right). \tag{17}$$

We obtain the following inequality from (17), by using equation (1) and taking modulus of a_2.

$$|a_2| \le \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{\left|(\mathfrak{q}+1)(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-2\mathfrak{q}\Theta_1^2\mathbf{Q}(\varkappa)\right|\mathfrak{q}}}.$$

Moreover, if we subtract (13) from (11), we obtain

$$2\mathfrak{q}(\mathfrak{q}+1)\Theta_2(a_3-a_2^2) = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left(m_2-n_2\right)+L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\left(m_1^2-n_1^2\right). \tag{18}$$

Then, in view of (14) and (15), (18) becomes

$$a_3 = \frac{L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa)}{2\mathfrak{q}^2\Theta_1^2}\left(m_1^2+n_1^2\right)+\frac{L_{\mathbf{P},\mathbf{Q},1}(\varkappa)}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\left(m_2-n_2\right).$$

Then, with the help of (1), we finally deduce

$$|a_3| \le \frac{\mathbf{P}^2(\varkappa)}{\mathfrak{q}^2\Theta_1^2}+\frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}.$$

■

Corollary 8 *If $\mathfrak{f}\in U^{\mu}_{\mathbf{\Sigma}}(\varkappa)$, then*

$$|a_2| \le \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{2\left|(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-\Theta_1^2\mathbf{Q}(\varkappa)\right|}}$$

and

$$|a_3| \le \frac{\mathbf{P}^2(\varkappa)}{\Theta_1^2}+\frac{|\mathbf{P}(\varkappa)|}{2\Theta_2}.$$

Corollary 9 *If $\mathfrak{f}\in S_{\mathbf{\Sigma}}(\varkappa)$, then*

$$|a_2| \le |\mathbf{P}(\varkappa)|\sqrt{\left|\frac{\mathbf{P}(\varkappa)}{2\mathbf{Q}(\varkappa)}\right|}$$

and

$$|a_3| \le \mathbf{P}^2(\varkappa)+\frac{|\mathbf{P}(\varkappa)|}{2}.$$

Corollary 10 *If $\mathfrak{f}\in C_{\mathbf{\Sigma}}(\varkappa)$, then*

$$|a_2| \le \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{2\left|\mathbf{P}^2(\varkappa)+4\mathbf{Q}(\varkappa)\right|}}$$

and

$$|a_3| \le \frac{\mathbf{P}^2(\varkappa)}{4}+\frac{|\mathbf{P}(\varkappa)|}{6}.$$

3 Fekete-Szegö Problem

The classical Fekete-Szegö inequality for the coefficients of $\mathfrak{f}$ selected from S is

$$\left|a_3 - \vartheta a_2^2\right| \leq 1 + 2\exp(-2\vartheta/(1-\vartheta)) \quad \text{for } \vartheta \in [0,1)\,.$$

As $\vartheta \to 1^-$, we have the elementary inequality $\left|a_3 - a_2^2\right| \leq 1$. Moreover, the coefficient functional

$$\digamma_\vartheta(\mathfrak{f}) = a_3 - \vartheta a_2^2$$

for the normalized analytic functions $\mathfrak{f}$ in the open unit disk Δ revives a major role in Geometric Function Theory. The problem of maximizing the absolute value of the functional $\digamma_\vartheta(\mathfrak{f})$ present the Fekete-Szegö problem (see, for details, [7]).

Next, for this section, we aim to provide Fekete-Szegö inequalities for functions in the class $U_{\mathbf{\Sigma}}^{\mu}(\mathfrak{q};\varkappa)$. These inequalities are given in the following theorem.

Theorem 11 *For $\vartheta \in \mathbb{R}$, let the function f given by (2) be in the class $U_{\mathbf{\Sigma}}^{\mu}(\mathfrak{q};\varkappa)$. Then*

$$\left|a_3 - \vartheta a_2^2\right| \leq \begin{cases} \frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}; & |\vartheta - 1| \leq \left|1 - \left(1 + \frac{2\mathfrak{q}\mathbf{Q}(\varkappa)}{(\mathfrak{q}+1)\mathbf{P}^2(\varkappa)}\right)\frac{\Theta_1^2}{\Theta_2}\right| \\ \frac{|1-\vartheta|\left|\mathbf{P}^3(\varkappa)\right|}{\mathfrak{q}\left|(\mathfrak{q}+1)(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-2\mathfrak{q}\Theta_1^2\mathbf{Q}(\varkappa)\right|}; & |\vartheta - 1| \geq \left|1 - \left(1 + \frac{2\mathfrak{q}\mathbf{Q}(\varkappa)}{(\mathfrak{q}+1)\mathbf{P}^2(\varkappa)}\right)\frac{\Theta_1^2}{\Theta_2}\right| \end{cases}.$$

Proof. From (17) and (18)

$$\begin{aligned} a_3 - \vartheta a_2^2 &= \frac{L^3_{\mathbf{P},\mathbf{Q},1}(\varkappa)\,(1-\vartheta)\,(m_2+n_2)}{2\mathfrak{q}\left\{\left[(\mathfrak{q}+1)\Theta_2 - \Theta_1^2\right] L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa) - \mathfrak{q}\Theta_1^2 L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\right\}} \\ &\quad + \frac{L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\,(m_2-n_2)}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2} \\ &= L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left[\left(\hbar(\vartheta;\varkappa) + \frac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\right)m_2 + \left(\hbar(\vartheta;\varkappa) - \frac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\right)n_2\right] \end{aligned}$$

where

$$\hbar(\vartheta;\varkappa) = \frac{L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa)\,(1-\vartheta)}{2\mathfrak{q}\left\{\left[(\mathfrak{q}+1)\Theta_2 - \Theta_1^2\right] L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa) - \mathfrak{q}\Theta_1^2 L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\right\}}.$$

In view of (1), we conclude that

$$\left|a_3 - \vartheta a_2^2\right| \leq \begin{cases} \dfrac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}, & 0 \leq |\hbar(\vartheta;\varkappa)| \leq \dfrac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2} \\ 2\,|\mathbf{P}(\varkappa)|\,|\hbar(\vartheta;\varkappa)|\,, & |\hbar(\vartheta;\varkappa)| \geq \dfrac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2} \end{cases}.$$

■

Corollary 12 *For $\vartheta \in \mathbb{R}$, let $\mathfrak{f}$ given by (2) be in the class $U_{\boldsymbol{\Sigma}}^{\mu}(\varkappa)$. Then*

$$\left|a_3 - \vartheta a_2^2\right| \leq \begin{cases} \dfrac{|\mathbf{P}(\varkappa)|}{2\Theta_2}; & |\vartheta - 1| \leq \left|1 - \left(1 + \frac{\mathbf{Q}(\varkappa)}{\mathbf{P}^2(\varkappa)}\right) \frac{\Theta_1^2}{\Theta_2}\right| \\ \dfrac{|1-\vartheta| \left|\mathbf{P}^3(\varkappa)\right|}{2\left|(\Theta_2 - \Theta_1^2)\mathbf{P}^2(\varkappa) - \Theta_1^2\mathbf{Q}(\varkappa)\right|}; & |\vartheta - 1| \geq \left|1 - \left(1 + \frac{\mathbf{Q}(\varkappa)}{\mathbf{P}^2(\varkappa)}\right) \frac{\Theta_1^2}{\Theta_2}\right| \end{cases}.$$

Corollary 13 *If $\mathfrak{f} \in S_{\boldsymbol{\Sigma}}(\varkappa)$, then*

$$\left|a_3 - \vartheta a_2^2\right| \leq \begin{cases} \dfrac{|\mathbf{P}(\varkappa)|}{2}, & |\vartheta - 1| \leq \dfrac{|\mathbf{Q}(\varkappa)|}{\mathbf{P}^2(\varkappa)} \\ \dfrac{|1-\vartheta| \left|\mathbf{P}^3(\varkappa)\right|}{2\left|\mathbf{Q}(\varkappa)\right|}, & |\vartheta - 1| \geq \dfrac{|\mathbf{Q}(\varkappa)|}{\mathbf{P}^2(\varkappa)} \end{cases}.$$

Corollary 14 *If $f \in C_{\boldsymbol{\Sigma}}(\varkappa)$, then*

$$\left|a_3 - \vartheta a_2^2\right| \leq \begin{cases} \dfrac{|\mathbf{P}(\varkappa)|}{6}, & |\vartheta - 1| \leq \dfrac{\left|\mathbf{P}^2(\varkappa) + 4\mathbf{Q}(\varkappa)\right|}{3\mathbf{P}^2(\varkappa)} \\ \dfrac{|1-\vartheta| \left|\mathbf{P}^3(\varkappa)\right|}{2\left|\mathbf{P}^2(\varkappa) + 4\mathbf{Q}(\varkappa)\right|}, & |\vartheta - 1| \geq \dfrac{\left|\mathbf{P}^2(\varkappa) + 4\mathbf{Q}(\varkappa)\right|}{3\mathbf{P}^2(\varkappa)} \end{cases}.$$

If we choose $\vartheta = 1$ in Theorem 11, we get the next corollary.

Corollary 15 *If $\mathfrak{f} \in U_{\boldsymbol{\Sigma}}^{\mu}(\mathfrak{q}; \varkappa)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}.$$

Corollary 16 *If $\mathfrak{f} \in U_{\boldsymbol{\Sigma}}^{\mu}(\varkappa)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{2\Theta_2}.$$

Corollary 17 *If $\mathfrak{f} \in S_{\boldsymbol{\Sigma}}(\varkappa)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{2}.$$

Corollary 18 *If $\mathfrak{f} \in C_{\boldsymbol{\Sigma}}(\varkappa)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{6}.$$

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