CERTAIN CLASSES OF MULTIVALENT FUNCTIONS DEFINED WITH HIGHER-ORDER DERIVATIVES

MOHAMED K. AOUF, ABDEL MONEIM LASHIN, AND TEODOR BULBOACĂ

Abstract. In this paper we derive some properties of multivalent functions belonging to the classes $R_{p,q}(\alpha)$, $B_{p,q}(\alpha)$ and $M_{p,q}(\alpha)$. The results obtained generalize the related works of some authors, and many other new results are obtained.

1. Introduction

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane, and let $\mathcal{A}_p$ denote the class of analytic and multivalent functions in $\mathbb{U}$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ z \in \mathbb{U} \quad (p \in \mathbb{N} := \{1, 2, \dots\}).$$

Also, let denote $\mathcal{A} := \mathcal{A}_1$.

For two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g, written as $f(z) \prec g(z)$, or simply $f \prec g$, if there exists a Schwarz function ω, that is ω is analytic $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$. If the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see [8, 15]).

For $0 \le \alpha < p - q$, $p > q$, $p \in \mathbb{N}$, and $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we say that $f \in \mathcal{A}_p$ is in the class $S^*_{p,q}(\alpha)$ if it satisfies the inequality

$$\operatorname{Re} \frac{z f^{(q+1)}(z)}{f^{(q)}(z)} > \alpha, \ z \in \mathbb{U}.$$

Also, we say that $f \in \mathcal{A}_p$ is in the class $K_{p,q}(\alpha)$ if the following inequality holds:

$$\operatorname{Re} \left[1 + \frac{z f^{(q+2)}(z)}{f^{(q+1)}(z)} \right] > \alpha, \ z \in \mathbb{U}.$$

The classes $S^*_{p,q}(\alpha)$ and $K_{p,q}(\alpha)$ were introduced and studied by Aouf [3, 5, 6], and we note that $S^*_{p,0}(\alpha) =: S^*_p(\alpha)$ and $K_{p,0}(\alpha) =: K_p(\alpha)$ are, respectively, the class of

2010 *Mathematics Subject Classification.* 30C45, 30C80.

Key words and phrases. Multivalent functions, p-valently starlike and convex functions, higher-order derivatives, differential subordinations, α-convex functions.

p-valently starlike functions of order α and the class of p-valently convex functions of order α $(0 \le \alpha < p)$ (see Owa [20] and Aouf [1, 2]).

Definition 1.1. For $0 \le \alpha < p - q$, $p > q$, $p \in \mathbb{N}$, and $q \in \mathbb{N}_0$, we say the function $f \in \mathcal{A}_p$ is in the class $C_{p,q}(\alpha)$ if there exists a function $g \in S^*_{p,q}(\alpha)$ such that

$$\operatorname{Re} \frac{z f^{(q+1)}(z)}{g^{(q)}(z)} > \alpha, \ z \in \mathbb{U}.$$

The class $C_{p,q}(\alpha)$ was introduced and studied by Aouf [4], and we note that $C_{p,0}(\alpha) =: C_p(\alpha)$ (see Aouf [7]).

Definition 1.2. Let $R_{p,q}(\alpha)$ be the subclass of $C_{p,q}(\alpha)$ obtained by choosing $g(z) = z^p$, that is the function $f \in \mathcal{A}_p$ belongs to the class $R_{p,q}(\alpha)$ if and only if it satisfies

$$\operatorname{Re} \frac{f^{(q+1)}(z)}{\delta(p,q) z^{p-q-1}} > \alpha, \ z \in \mathbb{U} \quad (0 \le \alpha < p - q), \tag{1.1}$$

where $\delta(p,q) = \dfrac{p!}{(p-q)!}$ $(p \ge q)$.

Remark 1.1. (i) It is easy to check that if the function $f \in \mathcal{A}_p$ satisfies the inequality

$$\left| \frac{f^{(q+1)}(z)}{z^{p-q-1}} - \delta(p, q+1) \right| < (p - q - \alpha)\delta(p,q), \ z \in \mathbb{U} \quad (0 \le \alpha < p - q), \tag{1.2}$$

then $f \in R_{p,q}(\alpha)$. Thus, if we denote by $S_{p,q}(\alpha)$ the class of functions $f \in \mathcal{A}_p$ that satisfies (1.2), then $S_{p,q}(\alpha) \subset R_{p,q}(\alpha)$.

(ii) We will denote by $B_{p,q}(\alpha)$ $(0 \le \alpha < \delta(p,q))$ the class $B_{p,q}(\alpha) := S_{p,q-1}\left(\frac{\alpha}{\delta(p,q-1)}\right)$. Therefore, the function $f \in \mathcal{A}_p$ belongs to the class $B_{p,q}(\alpha)$ $(0 \le \alpha < \delta(p,q))$ if and only if it satisfies

$$\left| \frac{f^{(q)}(z)}{z^{p-q}} - \delta(p,q) \right| < \delta(p,q) - \alpha, \ z \in \mathbb{U} \quad (0 \le \alpha < \delta(p,q)). \tag{1.3}$$

For $q := p - 1$ and $\beta := p!\alpha$, the inequality (1.2) reduces to

$$\left| f^{(p)}(z) - p! \right| < p! = \beta, \ z \in \mathbb{U} \quad (0 \le \beta < p!),$$

and the subclass $\mathbf{S}_p(\beta)$ of functions satisfying the above relation was introduced and studied by Saitoh [26]. Moreover, we remark the special cases $R_{p,0}(\alpha) =: R_p(\alpha)$ $(0 \le \alpha < p)$ (see Lee and Owa [11]) and $R_{1,0}(\alpha) =: R(\alpha)$ $(0 \le \alpha < 1)$ (see Owa et al. [23]). Also, the classes $R_{p,q-1}(\alpha)$ are connected with the results obtained by Saitoh in [27].

By using the differential higher order differential operators we define the following class of functions:

Definition 1.3. A function $f \in \mathcal{A}_p$ is said to be p-valently α-convex functions of higher order derivatives if it satisfies the inequality

$$\operatorname{Re}\left[(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right)\right] > 0,\ z \in \mathbb{U},$$

for some α $(\alpha \geq 0)$, and we will denote this class by $M_{p,q}(\alpha)$.

We note that $M_{p,q}(0) =: S^*_{p,q}(0)$ and $M_{p,q}(1) =: K_{p,q}(0)$. The class $M_{p,0}(\alpha) =: M_p(\alpha)$ was introduced and studied by Owa and Ren [24], and extends the class $M_{1,0}(\alpha) =: M(\alpha)$ defined by Mocanu [17] (see also, Mocanu and Reade [18], Miller [14], and Miller et al. [16]). Moreover, the class $M_{p,1-p}(\alpha) =: A(p,\alpha)$ was introduced and studied by Nunokawa [19], and subsequently studied by Fukui et al [9].

Definition 1.4. (i) Let $G(\alpha)$ be the class of functions g of the form

$$g(z) = 1 + \sum_{n=1}^{\infty} g_n z^n,\ z \in \mathbb{U}, \tag{1.4}$$

which are analytic in the unit disk $\mathbb{U}$ and satisfy

$$\operatorname{Re} g(z) > \alpha,\ z \in \mathbb{U},$$

for some α $(0 \leq \alpha < 1)$.

(ii) Further, let $G_b(\alpha)$ be the subclass of $G(\alpha)$ consisting of functions g of the form (1.4) and satisfying

$$g_1 = 2b(1-\alpha) = g'(0) \quad (0 \leq b \leq 1).$$

2. Preliminaries

In order to prove our main results we need the following lemmas.

Lemma 2.1. [10] *Let ω be regular in $\mathbb{U}$ with $\omega(0) = 0$. Then, if $|\omega|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, we have $z_0\omega(z_0) = k\omega(z_0)$, where $k \geq 1$.*

Lemma 2.2. [16] *If $f \in M(\alpha)$ $(\alpha \geq 0)$, then $f \in S^*(\beta(\alpha))$, where*

$$\beta(\alpha) := \begin{cases} 0, & \text{if } \ 0 \leq \alpha < 1, \\ \dfrac{\Gamma\left(\dfrac{1}{2} + \dfrac{1}{\alpha}\right)}{\sqrt{\pi}\,\Gamma\left(1 + \dfrac{1}{\alpha}\right)}, & \text{if } \ \alpha \geq 1. \end{cases} \tag{2.1}$$

The result is sharp.

Lemma 2.3. [17] *If* $f \in M(\alpha)$ $(\alpha \geq 0)$, *then* $f \in M(\beta)$ *for* $0 \leq \beta \leq \alpha$.

Lemma 2.4. [14] *If* $f \in M(\alpha)$ $(\alpha > 0)$, *then*

$$-K(\alpha, -r) \leq |f(z)| \leq K(\alpha, r),\ |z| = r,\ 0 < r < 1, \tag{2.2}$$

where

$$K(\alpha, r) := \left[\frac{1}{\alpha}\int_0^r t^{\frac{1}{\alpha}-1}(1-t)^{-\frac{2}{\alpha}}\,dt\right]^{\alpha}. \tag{2.3}$$

Equality holds in (2.2) *for the function* $f_\theta(\alpha, z)$ *given by*

$$f_\theta(\alpha, z) = \left[\frac{1}{\alpha}\int_0^z \zeta^{\frac{1}{\alpha}-1}(1-\zeta e^{i\theta})^{-\frac{2}{\alpha}}\,d\zeta\right]^{\alpha}, \tag{2.4}$$

where θ *is real and the powers appearing in* (2.3) *and* (2.4) *are meant as principal values.*

Lemma 2.5. [17] *The function* $f \in M(\alpha)$ $(\alpha > 0)$ *if and only if there exists a function* F *starlike in* $\mathbb{U}$, *such that*

$$f(z) = \left[\frac{1}{\alpha}\int_0^z \frac{(F(\zeta))^{\frac{1}{\alpha}}}{\zeta}\,d\zeta\right]^{\alpha},\ z \in \mathbb{U},$$

where the powers appearing in the formula are meant as principal values.

A function $f \in \mathcal{A}$ is said to be in the class $R(\alpha)$ if and only if it satisfies the inequality

$$\operatorname{Re} f'(z) > \alpha,\ z \in \mathbb{U},$$

for some α $(0 \leq \alpha < 1)$.

Lemma 2.6. [23] *If* $f \in R(\alpha)$ $(0 \leq \alpha < 1)$, *then*

$$\frac{f(z)}{z} \prec 2\alpha - 1 - \frac{2(1-\alpha)}{z}\log(1-z).$$

For $g \in G_b(\alpha)$, McCarty [12, 13] proved the next results:

Lemma 2.7. [12] *If* $g \in G_b(\alpha)$, *then*

$$\left|\frac{g'(z)}{g(z)}\right| \leq \frac{2(1-\alpha)}{1-r^2}\,\frac{b+2r+br^2}{1+2b(1-\alpha)r+(1-2\alpha)r^2},\ |z| = r,\ 0 < r < 1.$$

Lemma 2.8. [13] *If* $g \in G_b(\alpha)$*, then*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geq \begin{cases} \dfrac{-2(1-\alpha)r\,(b+2r+br^2)}{[1+2b\alpha r+(2\alpha-1)r^2]\,(1+2br+r^2)}, & \text{if } R' \leq R_b, \\ \dfrac{2\sqrt{\alpha A_1}-A_1-\alpha}{1-\alpha}, & \text{if } R' \geq R_b, \end{cases}$$

for $|z|=r$, $0<r<1$, *with* $R_b := A_b - D_b$, *where*

$$A_b := \frac{(1+br)^2-(2\alpha-1)(b+r)^2r^2}{(1-r^2)\,(1+2br+r^2)}, \quad D_b := \frac{2(1-\alpha)r(b+r)(1+br)r}{(1-r^2)\,(1+2br+r^2)},$$

and

$$R' := \sqrt{\alpha A_1}.$$

3. Some properties of the class $M_{p,q}(\alpha)$

The following result deals with an implication involving a similar relations that appear in the definition of the classes $R_{p,q}(\alpha)$ and $K_{p,q}(\alpha)$.

Theorem 3.1. *If the function* $f \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\left[\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} + \alpha\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right] > \alpha(p-q-1),\ z \in \mathbb{U},$$

for some α $(\alpha \geq 0)$ *and* $p>q$, *then*

$$\operatorname{Re}\frac{f^{(q+1)}(z)}{z^{p-q-1}} > \delta(p,q+1)\beta(\alpha),\ z \in \mathbb{U},$$

that is $f \in R_{p,q}\left((p-q)\beta(\alpha)\right)$, *where* $\beta(\alpha)$ *is given by* (2.1). *The result is sharp.*

Proof. Let define the function $g \in \mathcal{A}$ by

$$\frac{zg'(z)}{g(z)} = \frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}},\ z \in \mathbb{U}. \tag{3.1}$$

Differentiating logarithmically (3.1) with respect to z we obtain

$$\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - (p-q-1) = 1 + \frac{zg''(z)}{g'(z)} - \frac{zg'(z)}{g(z)},\ z \in \mathbb{U}, \tag{3.2}$$

and from (3.1) and (3.2) we have

$$\operatorname{Re}\left[\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} + \alpha\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \alpha(p-q-1)\right]$$
$$= \operatorname{Re}\left[(1-\alpha)\frac{zg'(z)}{g(z)} + \alpha\left(1+\frac{zg''(z)}{g'(z)}\right)\right] > 0,\ z \in \mathbb{U}.$$

This implies that $g \in M(\alpha)$, and by using Lemma 2.1 we have

$$\operatorname{Re}\frac{zg'(z)}{g(z)}=\operatorname{Re}\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}>\beta(\alpha),\ z\in\mathbb{U},$$

that is

$$\operatorname{Re}\frac{f^{(q+1)}(z)}{\delta(p,q)z^{p-q-1}}>(p-q)\beta(\alpha),\ z\in\mathbb{U},\tag{3.3}$$

where $\beta(\alpha)$ is given by (2.1). Since the result of Lemma 2.1 is sharp, the value $(p-q)\beta(\alpha)$ is the best lower bound for (3.3). □

For $q=0$, Theorem 3.1 reduced to the next result:

Corollary 3.2. *If the function $f\in\mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left[\frac{f'(z)}{pz^{p-1}}+\alpha\frac{zf''(z)}{f'(z)}\right]>\alpha(p-1),\ z\in\mathbb{U},$$

for some α $(\alpha\geq 0)$, then

$$\operatorname{Re}\frac{f'(z)}{z^{p-1}}>p\beta(\alpha),\ z\in\mathbb{U},$$

where $\beta(\alpha)$ given by (2.1). *The result is sharp.*

Remark 3.1. Putting $q=j-1$ $(1\leq j\leq p-1,\ p\in\mathbb{N})$ in Theorem 3.1 we get result obtained by Fukui et al. [9].

Theorem 3.3. *If $f\in M_{p,q}(\alpha)$ $(\alpha\geq 0)$, then $f\in S^*_{p,q}(\beta(\alpha;p,q))$, where*

$$\widetilde{\beta}(\alpha;p,q):=(p-q)\beta\left(\frac{\alpha}{p-q}\right)=\begin{cases}0, & \text{if } 0\leq\alpha<p-q,\\ \dfrac{(p-q)\Gamma\left(\dfrac{1}{2}+\dfrac{p-q}{\alpha}\right)}{\sqrt{\pi}\,\Gamma\left(1+\dfrac{p-q}{\alpha}\right)}, & \text{if } \alpha\geq p-q,\end{cases}$$

that is $M_{p,q}(\alpha)\subset S^*_{p,q}\left(\widetilde{\beta}(\alpha;p,q)\right)$. The result is sharp.

Proof. If $f\in M_{p,q}(\alpha)$ it follows that $f^{(q)}(z)\neq 0$ for all $z\in\mathbb{U}\setminus\{0\}$. For $f\in M_{p,q}(\alpha)$ let define the function $g\in\mathcal{A}$ by

$$g(z)=z\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\frac{1}{p-q}},\ z\in\mathbb{U},\tag{3.4}$$

where the power are meant as principal value. Differentiating logarithmically (3.4) with respect to z we get

$$\frac{zf^{(q+1)}(z)}{(p-q)f^{(q)}(z)}=\frac{zg'(z)}{g(z)},\ z\in\mathbb{U},\tag{3.5}$$

and

$$1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}=1+\frac{zg''(z)}{g'(z)}+(p-q-1)\frac{zg'(z)}{g(z)},\ z\in\mathbb{U}. \tag{3.6}$$

From (3.5) and (3.6) we deduce that

$$(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}+\alpha\left(1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right)=(p-q-\alpha)\frac{zg'(z)}{g(z)}+\alpha\left(1+\frac{zg''(z)}{g'(z)}\right)$$
$$=(p-q)\left[\left(1-\frac{\alpha}{p-q}\right)\frac{zg'(z)}{g(z)}+\frac{\alpha}{p-q}\left(1+\frac{zg''(z)}{g'(z)}\right)\right],\ z\in\mathbb{U},$$

hence

$$\frac{1}{p-q}\operatorname{Re}\left[(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}+\alpha\left(1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right)\right]$$
$$=\operatorname{Re}\left[\left(1-\frac{\alpha}{p-q}\right)\frac{zg'(z)}{g(z)}+\frac{\alpha}{p-q}\left(1+\frac{zg''(z)}{g'(z)}\right)\right],\ z\in\mathbb{U}.$$

This implies that $f\in M_{p,q}(\alpha)$ if and only if $g\in M\left(\frac{\alpha}{p-q}\right)$. Since $g\in M\left(\frac{\alpha}{p-q}\right)$, from Lemma 2.2 we get $g\in S^*\left(\beta\left(\frac{\alpha}{p-q}\right)\right)$, and according to (3.5) this last relation is equivalent to $f\in S^*_{p,q}\left((p-q)\beta\left(\frac{\alpha}{p-q}\right)\right)$, that is $f\in S^*_{p,q}\left(\widetilde{\beta}(\alpha;p,q)\right)$. Using the fact that the result of Lemma 2.2 is sharp, the bound $\widetilde{\beta}(\alpha;p,q)$ from the last relation is the best possible. □

For $\alpha=1$, Theorem 3.3 reduces to the next special case:

Corollary 3.4. *If $f\in K_{p,q}(0)$, then $f\in S^*_{p,q}\left(\widehat{\beta}(p,q)\right)$, where*

$$\widehat{\beta}(p,q):=\widetilde{\beta}(1;p,q).$$

that is $K_{p,q}(0)\subset S^*_{p,q}\left(\widehat{\beta}(p,q)\right)$. The result is sharp.

Theorem 3.5. *If $f\in M_{p,q}(\alpha)$ $(\alpha\geq 0)$, then $f\in M_{p,q}(\beta)$ for $0\leq\beta\leq\alpha$, that is*

$$M_{p,q}(\alpha)\subset M_{p,q}(\beta),\quad \textit{for}\quad 0\leq\beta\leq\alpha.$$

Proof. Like in the proof of Theorem 3.3, $f\in M_{p,q}(\alpha)$ $(\alpha\geq 0)$ if and only if $g\in M\left(\frac{\alpha}{p-q}\right)$, where the function g is given by (3.4). Since $0\leq\beta\leq\alpha$, according to Lemma 2.3 it follows that $g\in M\left(\frac{\beta}{p-q}\right)$, and this last relation is equivalent to $f\in M_{p,q}(\beta)$, which proves the assertion of Theorem 3.5. □

Theorem 3.6. *A function $f \in \mathcal{A}_p$ belongs to the class $M_{p,q}(\alpha)$ $(\alpha > 0)$ if and only if there exist a function $F \in S^* := S^*_{1,0}(0)$, such that*

$$f^{(q)}(z) = \delta(p,q) \left[\frac{p-q}{\alpha} \int_0^z \frac{(F(\zeta))^{\frac{p-q}{\alpha}}}{\zeta} \, d\zeta \right]^{\alpha}, \ z \in \mathbb{U}, \tag{3.7}$$

where the powers appearing in the formula are meant as principal values.

Proof. If we define the function g as in (3.4), from the proof of Theorem 3.3 we have that $f \in M_{p,q}(\alpha)$ $(\alpha \geq 0)$ if and only if $g \in M\left(\frac{\alpha}{p-q}\right)$. Then, from Lemma 2.5 we get that $g \in M\left(\frac{\alpha}{p-q}\right)$ if and only if there exists a function $F \in S^*$, such that

$$g(z) = \left[\frac{p-q}{\alpha} \int_0^z \frac{(F(\zeta))^{\frac{p-q}{\alpha}}}{\zeta} \, d\zeta \right]^{\frac{\alpha}{p-q}}, \ z \in \mathbb{U}.$$

Using the definition formula (3.4) we obtain that this last relation is equivalent to (3.7), which prove our result. □

Using the fact that $f \in M_{p,q}(\alpha)$ $(\alpha \geq 0)$ if and only if $g \in M\left(\frac{\alpha}{p-q}\right)$, where the function g is given by (3.4), from Lemma 2.4 we obatin the following theorem:

Theorem 3.7. *If $f \in M_{p,q}(\alpha)$ $(\alpha > 0)$, then*

$$-K_{p,q}(\alpha, -r) \leq \left| f^{(q)}(z) \right| \leq K_{p,q}(\alpha, r), \ |z| = r, \ 0 < r < 1, \tag{3.8}$$

where

$$K_{p,q}(\alpha, r) := \delta(p,q) \left[\frac{p-q}{\alpha} \int_0^z t^{\frac{p-q}{\alpha}-1} (1-t)^{\frac{-2(p-q)}{\alpha}} \, dt \right]^{\alpha}.$$

Equality holds in (3.8) *for*

$$f^{(q)}_{\theta;p,q}(\alpha, z) = \delta(p,q) \left[\frac{p-q}{\alpha} \int_0^z \zeta^{\frac{p-q}{\alpha}-1} (1-\zeta e^{i\theta})^{\frac{-2(p-q)}{\alpha}} \, d\zeta \right]^{\alpha},$$

where θ is real and all the powers appearing in the formulas are meant as principal values.

4. The subclass $R_{p.q}(\alpha)$

Theorem 4.1. *If* $f \in R_{p,q}\left(\frac{\alpha}{\delta(p,q)}\right)$ $(0 \le \alpha < \delta(p, q+1))$, *then*

$$\frac{1}{z}\int\limits_0^z \frac{f^{(q+1)}(\zeta)}{\zeta^{p-q-1}}\,d\zeta \prec 2\alpha - \delta(p,q+1) - \frac{2\left(\delta(p,q+1)-\alpha\right)}{z}\log(1-z). \tag{4.1}$$

Proof. If we define the function F by

$$F'(z) = \frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}} = 1 + c_1 z + c_2 z^2 + \dots,\ z \in \mathbb{U},$$

and $F(0) = 0$, then

$$F(z) = \frac{1}{\delta(p,q+1)}\int\limits_0^z \frac{f^{(q+1)}(\zeta)}{\zeta^{p-q-1}}\,d\zeta,\ z \in \mathbb{U}.$$

The fact that $f \in R_{p,q}\left(\frac{\alpha}{\delta(p,q)}\right)$ is equivalent to $f \in \mathcal{A}_p$ and

$$\operatorname{Re}\frac{f^{(q+1)}(z)}{z^{p-q-1}} > \alpha,\ z \in \mathbb{U} \quad (0 \le \alpha < \delta(p,q+1)). \tag{4.2}$$

From (4.2) it follows that

$$\operatorname{Re} F'(z) > \beta,\ z \in \mathbb{U} \quad \left(0 \le \beta < 1,\ \beta := \frac{\alpha}{\delta(p,q+1)}\right),$$

which, according to Lemma 2.6 implies

$$\frac{1}{\delta(p,q+1)z}\int\limits_0^z \frac{f^{(q+1)}(\zeta)}{\zeta^{p-q-1}}\,d\zeta \prec 2\beta - 1 - \frac{2(1-\beta)}{z}\log(1-z),$$

that is (4.1). □

For $q = 0$ in Theorem 4.1 we get the next special case:

Corollary 4.2. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\frac{f'(z)}{z^{p-1}} > \alpha,\ z \in \mathbb{U} \quad (0 \le \alpha < p),$$

then

$$\frac{1}{z}\int\limits_0^z \frac{f'(\zeta)}{\zeta^{p-1}}\,d\zeta \prec 2\alpha - p - \frac{2(p-\alpha)}{z}\log(1-z).$$

Remark 4.1. (i) Putting $q = j - 1$ $(1 \le j \le p)$ in Theorem 4.1 we get the result obtained by Owa [21, Theorem 1] and Saitoh [27, Theorem 5];

(ii) For $p = 1$ Corollary 4.2 reduces to the result of Owa et al. [23].

Putting $q = p - 1$ ($p \in \mathbb{N}$) in Theorem 4.1 we obtain the following corollary (see also Saitoh [25, Theorem 3]):

Corollary 4.3. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} f^{(p)}(z) > \alpha,\ z \in \mathbb{U} \quad (0 \le \alpha < p)$$

then

$$\frac{f^{(p-1)}(z)}{z} \prec 2\alpha - p! - \frac{2(p! - \alpha)}{z}\log(1 - z).$$

If we consider $p = 1$ in Corollary 4.3 we have the following corollary (see also Owa et al. [23] and Saitoh [25, Corollary 4]):

Corollary 4.4. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} f'(z) > \alpha,\ z \in \mathbb{U} \quad (0 \le \alpha < 1)$$

then

$$\frac{f(z)}{z} \prec 2\alpha - 1 - \frac{2(1 - \alpha)}{z}\log(1 - z).$$

Theorem 4.5. *If $f \in S_{p,q}(\alpha)$ and*

$$|\beta| \le \frac{\pi}{2} - \sin^{-1}\frac{p - q - \alpha}{p - q}, \tag{4.3}$$

then

$$\operatorname{Re}\left(e^{i\beta}\frac{f^{(q)}(z)}{z^{p-q}}\right) > 0,\ z \in \mathbb{U}.$$

where

Proof. From the definition of the class $S_{p,q}(\alpha)$ we have that $f \in S_{p,q}(\alpha)$ if and only if $f \in \mathcal{A}_p$ and (1.2) is satisfied. Using the fact that

$$|\zeta - \omega| < r,\ \zeta \in \mathbb{C} \quad (r < \omega) \quad \Rightarrow \quad |\arg\zeta| < \sin^{-1}\frac{r}{\omega},$$

from (1.2) we obtain

$$\left|\arg\frac{f^{(q+1)}(z)}{z^{p-q-1}}\right| < \sin^{-1}\frac{(p - q - \alpha)\delta(p, q)}{\delta(p, q + 1)} = \sin^{-1}\frac{p - q - \alpha}{p - q},\ z \in \mathbb{U}. \tag{4.4}$$

From (4.3) and (4.4) it follows that

$$\left|\arg\left(e^{i\beta}\frac{f^{(q+1)}(z)}{z^{p-q-1}}\right)\right| \le |\beta| + \left|\arg\frac{f^{(q+1)}(z)}{z^{p-q-1}}\right| < \frac{\pi}{2},\ z \in \mathbb{U},$$

that is

$$\operatorname{Re}\left(e^{i\beta}\frac{f^{(q+1)}(z)}{z^{p-q-1}}\right) > 0,\ z \in \mathbb{U}. \tag{4.5}$$

If we define the function ω by

$$e^{i\beta}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} - i\sin\beta = \cos\beta\frac{1+\omega(z)}{1-\omega(z)},\ z \in \mathbb{U}, \tag{4.6}$$

with $\omega(z) \neq 1$ for all $z \in \mathbb{U}$, we see that ω is analytic in $\mathbb{U}$ and $\omega(0) = 0$. It follows that

$$e^{i\beta}f^{(q)}(z) - i\delta(p,q)\sin\beta z^{p-q} = \delta(p,q)\cos\beta\frac{1+\omega(z)}{1-\omega(z)}z^{p-q},\ z \in \mathbb{U},$$

and differentiating the above relation with respect to z we obtain

$$e^{i\beta}f^{(q+1)}(z) - i\delta(p,q+1)\sin\beta z^{p-q-1}$$
$$= \delta(p,q)\cos\beta\left[(p-q)z^{p-q-1}\frac{1+\omega(z)}{1-\omega(z)} + z^{p-q-1}\frac{2z\omega'(z)}{(1-\omega(z))^2}\right],\ z \in \mathbb{U},$$

therefore

$$e^{i\beta}\frac{f^{(q+1)}(z)}{z^{p-q-1}} - i\delta(p,q+1)\sin\beta = \delta(p,q)\cos\beta\left[(p-q)\frac{1+\omega(z)}{1-\omega(z)} + \frac{2z\omega'(z)}{(1-\omega(z))^2}\right],\ z \in \mathbb{U}.$$

If we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\le|z_0|}|\omega(z)| = |\omega(z_0)| = 1,$$

then $\omega(z_0) = e^{i\theta}$ for some $\theta \in (0, 2\pi)$. Since $\cos\beta > 0$, by using Lemma 2.1 we get

$$\operatorname{Re}\left(e^{i\beta}\frac{f^{(q+1)}(z_0)}{z^{p-q-1}}\right) = \operatorname{Re}\left[e^{i\beta}\frac{f^{(q+1)}(z_0)}{z_0^{p-q-1}} - i\delta(p,q+1)\sin\beta\right]$$
$$= \delta(p,q)\cos\beta\operatorname{Re}\left[(p-q)\frac{1+e^{i\theta}}{1-e^{i\theta}} + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right] = \delta(p,q)\cos\beta\frac{k}{\cos\theta-1} < 0,$$

where $k \geq 1$. The above inequality contradicts (4.5), therefore $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. From (4.6), since $\cos\beta > 0$, we conclude that

$$\operatorname{Re}\left(e^{i\beta}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right) = \operatorname{Re}\left(e^{i\beta}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} - i\sin\beta\right) > 0,\ z \in \mathbb{U}.$$

□

Putting $q = 0$ in Theorem 4.5 we have:

Corollary 4.6. *If* $f \in S_{p,0}(\alpha)$ *and*

$$|\beta| \le \frac{\pi}{2} - \sin^{-1}\frac{p-\alpha}{p},$$

then

$$\operatorname{Re}\left(e^{i\beta}\frac{f(z)}{z^p}\right) > 0,\ z \in \mathbb{U}.$$

Remark 4.2. We note that the result of Corollary 4.6 for $p = 1$ was obtained by Owa et al. [22].

If we take $q = j - 1$ $(1 \le j \le p)$ in Theorem 4.5, we deduce the next result:

Corollary 4.7. *If* $f \in S_{p,j-1}(\alpha)$ $(1 \le j \le p)$ *and*

$$|\beta| \le \frac{\pi}{2} - \sin^{-1}\frac{p-j+1-\alpha}{p-j+1},$$

then

$$\operatorname{Re}\left(e^{i\beta}\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right) > 0,\ z \in \mathbb{U}.$$

Remark 4.3. Our result in Corollary 4.7 correct the result obtained by Owa [21, Theorem 3].

We will add at the end of this section the following inclusion theorem:

Theorem 4.8. *If* $f \in R_{p,q}(\alpha)$, *then* $f \in R_{p,q-1}\left(\widehat{\beta}\right)$ $(1 \le q < p)$, *where*

$$\widehat{\beta} = \frac{\alpha(p-q+1)}{p-q}, \tag{4.7}$$

that is $R_{p,q}(\alpha) \subset R_{p,q-1}\left(\frac{\alpha(p-q+1)}{p-q}\right)$.

Proof. For the function $f \in \mathcal{A}_p$, according to the inequality (1.1) we have

$$f \in R_{p,q}(\alpha) \Leftrightarrow \operatorname{Re}\left[\frac{f^{(q+1)}(z)}{\delta(p,q)z^{p-q-1}} - \alpha\right] > 0,\ z \in \mathbb{U} \quad (0 \le \alpha < p-q). \tag{4.8}$$

We will determine the biggest value of $\widehat{\beta} \in \mathbb{R}$, such that $f \in R_{p,q-1}\left(\widehat{\beta}\right)$, that is

$$\operatorname{Re}\left[\frac{f^{(q)}(z)}{\delta(p,q-1)z^{p-q}} - \widehat{\beta}\right] > 0,\ z \in \mathbb{U}.$$

Let define the function w, analytic in $\mathbb{U}$, with $w(0) = 0$ and $w(z) \neq 1$ for all $z \in \mathbb{U}$, such that

$$\frac{f^{(q)}(z)}{\delta(p,q-1)z^{p-q}} - \beta = \left(p-q+1-\widehat{\beta}\right)\frac{1+w(z)}{1-w(z)},\ z \in \mathbb{U}. \tag{4.9}$$

Differentiating the above relation we get

$$\frac{f^{(q+1)}(z)}{\delta(p,q)z^{p-q-1}}-\alpha=-\alpha+\frac{\widehat{\beta}(p-q)}{p-q+1}$$
$$+\frac{p-q+1-\widehat{\beta}}{p-q+1}\left[(p-q)\frac{1+w(z)}{1-w(z)}+\frac{2zw'(z)}{(1-w(z))^2}\right],\ z\in\mathbb{U}.$$

Supposing that there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1,$$

by using Lemma 2.1, and letting $w(z_0)=e^{i\theta}$ for some $\theta\in(0,2\pi)$, we get

$$\frac{f^{(q+1)}(z_0)}{\delta(p,q)z_0^{p-q-1}}-\alpha=-\alpha+\frac{\widehat{\beta}(p-q)}{p-q+1}+\frac{p-q+1-\widehat{\beta}}{p-q+1}\left[(p-q)\frac{1+e^{i\theta}}{1-e^{i\theta}}+\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right],$$

and therefore

$$\operatorname{Re}\left[\frac{f^{(q+1)}(z_0)}{\delta(p,q)z_0^{p-q-1}}-\alpha\right]=-\alpha+\frac{\widehat{\beta}(p-q)}{p-q+1}$$
$$+\frac{p-q+1-\widehat{\beta}}{p-q+1}\operatorname{Re}\left[(p-q)\frac{1+e^{i\theta}}{1-e^{i\theta}}+\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right].$$

This last relation is equivalent to

$$\operatorname{Re}\left[\frac{f^{(q+1)}(z_0)}{\delta(p,q)z_0^{p-q-1}}-\alpha\right]=-\alpha+\frac{\widehat{\beta}(p-q)}{p-q+1}+\frac{p-q+1-\widehat{\beta}}{p-q+1}\left(-\frac{2}{4\sin^2\frac{\theta}{2}}\right),$$

and assuming that $\widehat{\beta}\leq p-q+1$, from the above identity we deduce that

$$\operatorname{Re}\left[\frac{f^{(q+1)}(z_0)}{\delta(p,q)z_0^{p-q-1}}-\alpha\right]\leq-\alpha+\frac{\widehat{\beta}(p-q)}{p-q+1}=0,$$

if $\widehat{\beta}$ is given by (4.7). Moreover, this value of $\widehat{\beta}$ satisfies the inequality $\widehat{\beta}<p-q+1$, and therefore the above inequality contradicts the assumption (4.8).

It follows that $|w(z)|<1$ for all $z\in\mathbb{U}$, and using the fact that $\widehat{\beta}<p-q+1$, from (4.9) we obtain our conclusion. □

5. The subclass $B_{p,q}(b,\alpha)$

Let $B_{p,q}(b,\alpha)$ be the subclass of $B_{p,q}(\alpha)$ consisting of functions $f\in B_{p,q}(\alpha)$ satisfying

$$a_{p+1}=2b\left(\delta(p,q)-\alpha\right)\frac{(p-q+1)!}{(p+1)!},\quad(p>q,\ 0\leq\alpha<\delta(p,q),\ 0\leq b\leq1).$$

For $f\in B_{p,q}(\alpha)$ we proved the next result:

Theorem 5.1. *If* $f \in B_{p,q}(b,\alpha)$, *then*

$$\left|\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right| \leq (p-q)$$
$$+\frac{2\left(\delta(p,q)-\alpha\right)r}{1-r^2}\frac{b+2r+br^2}{\delta(p,q)+2b\left(\delta(p,q)-\alpha\right)r+\left(\delta(p,q)-2\alpha\right)r^2}, \tag{5.1}$$

where $|z|=r,\ 0<r<1$.

Proof. If $f \in B_{p,q}(b,\alpha)$, then

$$f(z)=z^p+2b\left(\delta(p,q)-\alpha\right)\frac{(p-q+1)!}{(p+1)!}z^{p+1}+\ldots,\ z\in\mathbb{U},$$

and we obtain that

$$\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}=1+2b\left(1-\frac{\alpha}{\delta(p,q)}\right)z+\ldots,\ z\in\mathbb{U}, \tag{5.2}$$

with $0\leq\dfrac{\alpha}{\delta(p,q)}<1$ and $0\leq b\leq 1$. Since $f\in B_{p,q}(b,\alpha)$, from (1.3) and (5.2) it follows that

$$\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\in G_b\left(\frac{\alpha}{\delta(p,q)}\right).$$

Using Lemma 2.7 for the function $\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}$ we conclude that

$$\left|\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}-(p-q)\right|\leq$$
$$\frac{2(\delta(p,q)-\alpha)r}{1-r^2}\frac{b+2r+br^2}{\delta(p,q)+2b(\delta(p,q)-\alpha)r+\left(\delta(p,q)-2\alpha\right)r^2},\ |z|=r,\ 0<r<1,$$

which implies the conclusion (5.1). □

For $q=0$ Theorem 5.1 reduces to the next special case:

Corollary 5.2. *If* $f\in B_{p,0}(b,\alpha)$, *then*

$$\left|\frac{zf'(z)}{f(z)}\right|\leq p+\frac{2(1-\alpha)r}{1-r^2}\frac{b+2r+br^2}{1+2b(1-\alpha)r+(1-2\alpha)r^2},\ |z|=r,\ 0<r<1.$$

With a similar proof like for Theorem 5.1, using Lemma 2.8 we obtain the following theorem:

Theorem 5.3. *If $f \in B_{p,q}(b,\alpha)$, then*

$$\operatorname{Re}\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \geq \begin{cases} (p-q) - \dfrac{2\left(\delta(p,q)-\alpha\right)r(b+2r+br^2)}{\left(\delta(p,q)+2b\alpha r+(2\alpha-\delta(p,q))r^2\right)\left(1+2br+r^2\right)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad R' \leq R_b, \\ (p-q) + \dfrac{2\sqrt{\delta(p,q)\alpha M_1} - \delta(p,q)M_1 - \alpha}{\delta(p,q)-\alpha}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad R' \geq R_b, \end{cases}$$

for $|z| = r$, $0 < r < 1$, with $R_b := M_b - N_b$, where

$$M_b := \frac{\delta(p,q)(1+br)^2 - (2\alpha-\delta(p,q))(b+r)^2r^2}{\delta(p,q)(1-r^2)\left(1+2br+r^2\right)},$$

$$N_b := \frac{2\left(\delta(p,q)-\alpha\right)r(b+r)(1+br)r}{\delta(p,q)(1-r^2)\left(1+2br+r^2\right)},$$

and

$$R' := \sqrt{\frac{\alpha}{\delta(p,q)}M_1}.$$

Taking $q = 0$ in Theorem 5.3 we obtain the next special case:

Corollary 5.4. *If $f \in B_{p,0}(b,\alpha)$, then*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \begin{cases} p - \dfrac{2(1-\alpha)r(b+2r+br^2)}{\left(1+2b\alpha r+(2\alpha-1)r^2\right)\left(1+2br+r^2\right)}, & \text{if} \quad R' \leq R_b, \\ p + \dfrac{2\sqrt{\alpha M_1} - M_1 - \alpha}{1-\alpha}, & \text{if} \quad R' \geq R_b, \end{cases}$$

for $|z| = r$, $0 < r < 1$, with $R_b := M_b - N_b$, where

$$M_b := \frac{(1+br)^2 - (2\alpha-1)(b+r)^2r^2}{(1-r^2)\left(1+2br+r^2\right)}, \quad N_b := \frac{2(1-\alpha)r(b+r)(1+br)r}{(1-r^2)\left(1+2br+r^2\right)},$$

and

$$R' := \sqrt{\alpha M_1}.$$

Remark 5.1. (i) Putting $q = j$ $(1 \leq j \leq p)$ in Theorems 5.1 and 5.3 we get the results obtained by Owa [21, Theorems 5 and 6];

(ii) For $p = 1$ the Corollaries 5.2 and 5.4 reduce to the results of by McCarty [12, 13].

Acknowledgments. The authors are grateful to the reviewers of this article, that gave valuable remarks, comments, and advices, in order to revise and improve the results of the paper.

References

[1] Aouf MK. On a class of p-valent starlike functions of order α. Internat. J. Math. Math. Sci., 10(1987) no. 4, 733-744

[2] Aouf MK. A generalization of mulivalent functions with negative coefficients. J. Korean Math. Soc., 25(1988), no. 1, 53-66.

[3] Aouf MK. Certain classes of multivalent functions with negative coefficients defined by using a differential operator. J. Math. Appl., 30(2008), 5-21.

[4] Aouf MK. Certain subclasses of p-valent starlike functions defined by using a differential operator. Appl. Math. Comput., 206(2008), 867-875.

[5] Aouf MK. Some families of p-valent functions with negative coefficients. Acta Math. Univ. Comenian. (N.S.), 78(2009), no. 1, 121-135.

[6] Aouf MK. Bounded p-valent Robertson functions defined by using a differential operator. J. Franklin Inst., 347(2010), 1972-1941.

[7] Aouf MK. Some inclusion relationships associated with Dizok-Srivastava operator. Appl. Math. Comput., 216(2010), 431-437.

[8] Bulboacă T. Differential Subordinations and Superordinations. New results. House of Scientific Book Publ., Cluj-Napoca, 2005.

[9] Fukui S, Ren F, Owa S, Nunokawa M. On certain multivalent functions. Bull. Fac. Edu. Wakayama Univ. Nat. Sci., 38(1989), 5-8.

[10] Jack IS. Functions starlike and convex of order α. J. Lond. Math. Soc. (2), 2(1971), no. 3, 469-474.

[11] Lee SK, Owa S. A subclass of p-valently close to convex functions of order α. Appl. Math. Lett., 5(1992), no. 5, 3-6.

[12] McCarty CP. Functions with real part greater than α. Proc. Amer. Math. Soc., 35(1972), 211-216.

[13] McCarty CP. Two radius of convexity problems. Proc. Amer. Math. Soc., 42 (1974), 153-160.

[14] Miller SS. Distortion properties of alpha-starlike functions. Proc. Amer. Math. Soc., 38(1973), 311-318.

[15] Miller SS, Mocanu PT. Differential Subordinations. Theory and Applications. Series on Monographs and Textbooks in Pure and Appl. Math. no. 255, Marcel Dekker, Inc., New York, 2000.

[16] Miller SS, Mocanu PT, Reade MO. The order of starlikeness of alpha-convex functions. Mathematica (Cluj), 20(1978), 25-30.

[17] Mocanu PT. Une propriété de convexité generaliseé dans la théorie de la représentation conforme. Mathematica (Cluj), 11(1969), 127-133.

[18] Mocanu PT, Reade MO. On generalized convexity in conformal mappings. Rev. Roum. Math. Pures Appl., 46(1971), 1541-1544.

[19] Nunokawa M. On the theory of multivalent functions. Tsukuba J. Math., 11(1987), no. 2, 273-286.

[20] Owa S. On certain classes of p-valent functions with negative coefficients. Bull. Belg. Math. Soc. Simon Stevin, 59(1985), no. 4, 385-402.

[21] Owa S. Some properties of certain multivalently functions. Math. Nachr., 155(1992), 167-185.

[22] Owa S, Aouf MK, Nasr MA. Note on certain subclass of close-to-convex functions of order α. Internat. J. Math. Math. Sci., 13(1990), no. 1, 189-192.

[23] Owa S, Ma W, Liu L. On a class of analytic functions satisfying $\mathrm{Re}(f'(z)) > \alpha$. Bull. Korean Math. Soc., 25(1988), 211-224.

[24] Owa S, Ren F. On a class of p-valently α-convex functions. Math. Nachr., 146(1990), 17-21.

[25] Saitoh H. Some properties of certain analytic functions. Topics in Univalent Functions and Its Applications, 714(1990), 160-167.

[26] Saitoh H. Some properties of certain multivalent functions. Tsukuba J. Math., 15(1991), no.1, 105-111.

[27] Saitoh H. On certain class of multivalent functions. Math. Japon., 37(1992), no.5, 871-875.

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Email address: mkaouf127@yahoo.com

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Email address: ylashin@mans.edu.eg

Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

Email address: bulboaca@math.ubbcluj.ro