Plane wave diffraction by the strip with an integral boundary condition

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Abstract: In this article, a new solution method is proposed for the plane wave diffraction by a strip. On the surface of the strip, an integral boundary condition is used. The impedance of the strip is investigated. The theoretical and the numerical analyzes showed that there is a relation between the complex-valued fractional order of the integral boundary condition and the properties of the material such as the impedance. As a further study, the total radar cross section is investigated using the proposed method.

Key words: electromagnetic diffraction, fractional boundary condition, fractional strip, fractional calculus

1. Introduction

Initial studies about the application of the fractional approach to the electromagnetic theory are performed by Engheta in the 1990s [1–3]. He introduced the idea of "fractional paradigm in electromagnetics" which states that there are continuous intermediate stages between two canonical states of the electromagnetic field. Using the properties of the fractional derivative approach, the intermediate stages of fields or sources between two canonical states can be described [1–5]. Since then, several studies have been performed on scattering problems [6–10]. The integral boundary condition which corresponds to an intermediate boundary condition between Dirichlet and Neumann boundary conditions is used in order to explain the scattering properties of different geometries. By determining the fractional order (FO), the scattering properties of different materials are investigated. Throughout the study, the integral boundary condition also can be pronounced as the fractional boundary condition (FBC) which was explained in previous studies [7, 8].

The boundary conditions used in previous works [7–9] describe a new material property (Perfect Electric Conducting (PEC), Perfect Magnetic Conducting (PMC) or in between). In these studies, theoretical and numerical results are obtained for the plane wave diffraction by one strip, two strips with the same length, and the semi-infinite strip. In the case of the real fractional order, the impedance of the strip is found as purely imaginary value.

In this paper, the complex fractional order that has both real and imaginary parts is introduced for the first time. The impedance of the strip is found to be complex-valued. Therefore, the fractional boundary condition with the complex fractional order describes a new kind of material with different properties. Several different cases such as different operating frequencies, strip length and the fractional orders are investigated

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both theoretically and numerically. Then, the results are presented.

This paper is organized as follows. In Section 2, the formulation of the problem and the theoretical results are introduced briefly. In Section 3, the convergence study for the numerical analysis is presented. Next, in Sections 4 and 5, Physical Characteristics of the Electric Field and Numerical Results are introduced, respectively. Finally, the conclusion of this paper is drawn in Section 6.

2. Formulation of the Problem

The main focus of this section is the investigation of E-polarized electromagnetic plane wave diffraction by a strip with the fractional boundary condition (FBC). There is a strip which has width 2a and infinitesimal height. The strip is located at y = 0 and has an infinite length in the z-axis. The geometry of the problem is shown in Figure 1. From the definition of Riemann-Liouville equation [11], the fractional derivative can be found as

\[
D_y^\nu f(y) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dy} \int_{-\infty}^{y} \frac{f(t)}{(y-t)^\nu} dt
\]  

(1)

Here, \(D_y^\nu\) stands for the derivative operator and means that the derivative is taken with respect to \(y\) in the order of \(\nu\). The real part of the fractional order is shown as \(\text{Re}(\nu)\) which varies within the range of \(0 < \text{Re}(\nu) < 1\), and \(\Gamma(\nu)\) is the Gamma function.

After defining the fractional derivative, it is needed to define the fractional boundary condition (FBC) [7–10]. \(U(x,y)\) is the function subjecting to the FBC at \(y = d\) which is the boundary of the strip in a two-dimensional case. For the plane surface boundary \(S\) located at \(y = d\), the general form of FBC is written below as follows

\[
D_y^\nu U(x,y)|_{y \in S} = 0, y = \pm d
\]  

(2)

In electromagnetic problems, \(U(x,y)\) becomes \(E_z(x,y)\) or \(H_z(x,y)\) depending on the polarization. In this problem, the incident wave is E-polarized, therefore \(U(x,y)\) stands for \(E_z(x,y)\). The total electric field \(\vec{E}_z\) is composed of the superposition of two fields which are a plane wave as an incidence wave \(\vec{E}_i\) and the scattered electric field \(\vec{E}_s\), respectively. The monochromatic incidence wave can be denoted as

\[
\vec{E}_i = \vec{a}_z e^{-ik(x \cos \theta + y \sin \theta)}
\]  

(3)

Here, \(\vec{a}_z\) is the unit vector along the z-axis, \(\theta\) is the angle of incidence, \(k = \frac{2\pi}{\lambda}\) is the wave number, \(\lambda\) is the wavelength in the free space. Note that, the time dependency throughout this study is taken as \(e^{-i\omega t}\) and then, is omitted. The total electric field can be represented as follows:

\[
\vec{E}_z = \vec{E}_i + \vec{E}_s
\]  

(4)

In order to find the scattered electric field, the total electric field needs to satisfy the boundary condition on the surface of the strip [7–10]. Here, the boundary condition is the fractional boundary condition and the mathematical expression is given as

\[
D_{ky}^\nu E_z(x,y)|_{y=\pm 0} = 0
\]  

(5)
where $x, -a < x < a$.

Here, the derivative is taken with respect to $ky$ which is dimensionless parameter and $y$ is the normal direction on the strip. The boundary condition given in (5) covers both Dirichlet and Neumann type boundary conditions for $\nu = 0$ and $\nu = 1$, respectively. For fractional order, $\nu$ is equal to 0, it corresponds to perfect electric conducting (PEC) surface, whereas, for fractional order, $\nu$ is equal to 1, the surface corresponds to perfect magnetic conducting (PMC) surface. Therefore, FBC is a more general boundary condition covering both Dirichlet and Neumann type boundary conditions [7–10].

The scattered electric field can be expressed as the convolution of fractional current density induced on the strip and fractional Green’s function as given as

$$ E_z^s(x, y) = \int_{-\infty}^{\infty} f^{1-\nu}(x')G^{\nu}(x-x', y)dx' $$

(6)

Here, $f^{1-\nu}(x')$ is an unknown function, which is called the fractional current density and has only non-zero values on the strip. The current density has zero values outside of the strip. The fractional Green’s function $G^{\nu}(x)$ has the form given as follows [5–7]:

$$ G^{\nu}(x-x', y) = -\frac{i}{4}D_{ky}\text{H}_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) $$

(7)

where $\text{H}_0^{(1)}$ is the Hankel Function of the first kind and zero-order. Its spectral representation is given in (8). By the spectral representation, the Hankel function can be written as the summation of plane waves [5–7].

$$ \text{H}_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ik(x-x')\alpha + |y|\sqrt{1-\alpha^2}} \frac{d\alpha}{\sqrt{1-\alpha^2}} $$

(8)

In order to find the expression for the scattered electric field, (7) is inserted into (6) via using the equation given in (8). Then, the scattered electric field is found as (9). It should be denoted that the fractional derivative...
of the exponential term is taken as $D_x^\nu e^{ikx} = (ik)^\nu e^{ikx}$ which is found by using (1). [11, 12]. The reason of denoting the derivative of exponents is that both the incidence and the scattered electric field can be expressed in terms of exponentials and then, FBC can be applied.

$$E_z(x, y) = -ie^{\pm i\nu \pi/4} \int_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{i\kappa[(x\alpha \pm y\sqrt{1-\alpha^2})(1-\alpha^2)^{\nu-1/2}] d\alpha$$  (9)

where,

$$F^{1-\nu}(\alpha) = \int_{-1}^{1} \widetilde{f}^{1-\nu}(\xi) e^{-i\alpha\xi} d\xi, \quad \widetilde{f}^{1-\nu}(\xi) = \alpha f^{1-\nu}(\alpha\xi),$$

$$\epsilon = ka, \quad \xi = \frac{\epsilon}{\alpha}, \quad \widetilde{f}^{1-\nu}(\xi) = \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{-i\alpha\xi} d\alpha.$$  (10)

The integral representation in (9) satisfies the wave equation and the Sommerfeld radiation condition where (+) sign stands for the upper half space ($y > 0$) and (-) sign corresponds to the lower half space ($y < 0$). After finding the scattered electric field, the FBC is applied to the total electric field as given in (5). Then, the integral equation (IE) is obtained as given in (10).

$$\int_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{i\epsilon\alpha} (1-\alpha^2)^{\nu-1/2} d\alpha = -4i\pi e^{-i\nu\pi/2} (\sin \theta)^\nu e^{-ikx \cos \theta}$$  (11)

By multiplying the integral equation in (10) by $e^{-i\varepsilon \alpha \beta}$ and integrating with respect to x from -a to a, IE (11) is obtained. For any arbitrary fractional order $\nu$, IE (11) needs to be solved in order to find Fourier transform of the fractional current density.

$$\int_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{i\epsilon\alpha} (1-\alpha^2)^{\nu-1/2} d\alpha = -4i\pi e^{-i\nu\pi/2} (\sin \theta)^\nu e^{-ikx \cos \theta}$$  (12)

In order to satisfy Meixner’s edge condition [13], series of Gegenbauer polynomials representation $\tilde{C}_n^\nu(\xi)$ for the normalized current density $\tilde{f}^{1-\nu}$ with the weighting function $(1-\xi)^{\nu-1/2}$ and the unknown $\zeta_n^\alpha$ are used. The normalized current density can be expressed as (12).

$$\tilde{f}^{1-\nu}(\xi) = (1-\xi)^{\nu-1/2} \sum_{n=0}^{\infty} \zeta_n^\alpha \frac{C_n^\nu(\xi)}{\alpha}$$  (13)

Then, Fourier transform of the current density can be found as (13) [7–10].

$$F^{1-\nu}(\alpha) = \frac{2\pi}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} (-i)^n \zeta_n^\alpha \beta_n^{\nu} J_{\nu+\nu}(\epsilon\alpha)$$  (14)

Here, $\epsilon = ka$, $J_{\nu+\nu}(\epsilon\alpha)$ are Bessel functions and $\beta_n^{\nu} = \Gamma(n+2\nu)\Gamma(n+1)$.  (15)
By substituting (13) into (11), the integral equation (11) is converted into the system of linear algebraic equation (SLAE) by introducing the unknown $\zeta^\alpha_n$. Note that, the properties of discontinuous integrals of Weber-Shafheitlin and (14) are taken into account [14, 15]. SLAE can be given as (15).

$$\int_{-\infty}^{\infty} \frac{J_{n+\nu}(\varepsilon \alpha \sin(\varepsilon(\alpha - b)))}{\alpha - \beta} d\alpha = \frac{J_{n+\nu}(\varepsilon \beta)}{\beta^\nu},$$  \hspace{1cm} (14)$$

$$\sum_{n=0}^{\infty} (-i)^n \zeta^\alpha_n \beta^\nu C^\nu_{mn} = \chi^\nu_m$$  \hspace{1cm} (15)

where,

$$C^\nu_{mn} = \int_{-\infty}^{\infty} J_{n+\nu}(\varepsilon \alpha) J_{m+\nu}(\varepsilon \alpha) \left(1 - \alpha^2\right)^{-\frac{1}{2}} \frac{d\alpha}{\alpha^{2\nu}},$$

$$\chi^\nu_m = -2i \Gamma(\nu + 1)(-i)^\nu (2\varepsilon^\nu \sin \theta)^\nu \frac{J_{m+\nu}(\varepsilon \cos \theta)}{(\cos \theta)^\nu}.$$

3. Investigation of Convergence in Numerical Analysis

When FO, $\nu$ becomes the complex-valued, the convergence of $C^\nu_{mn}$ given in (15) is not satisfactory. In order to have a better convergence, the manipulation in the integral expression of $C^\nu_{mn}$ is done. In other words, the integral $C^\nu_{mn}$ is split into two parts as the first integral from $-\infty$ to 0 and the second integral from 0 to $\infty$. Then, in the first integral part, the change of variable is applied ($q \rightarrow -q$). After that, the expression given in (16) is found.

$$C^\nu_{mn} = \left[1 + (-1)^{m+n}\right] \int_{0}^{\infty} J_{n+\nu}(\varepsilon q) J_{m+\nu}(\varepsilon q) \left(1 - q^2\right)^{-\frac{1}{2}} \frac{dq}{q^{2\nu}},$$  \hspace{1cm} (16)

In order to increase the convergence, the term $(1 - q^2)^{\nu-1/2}$ in (16) can be written as (17).

$$\left(1 - q^2\right)^{\nu-\frac{1}{2}} = \left(1 + (iq)^2\right)^{\nu-\frac{1}{2}} = q^{2\nu-1} \left[-\left(1 - \frac{1}{q^2}\right)\right]^{\nu-\frac{1}{2}}.$$  \hspace{1cm} (17)

Here, we defined $\gamma_\nu(q)$ as $\gamma_\nu(q) = \left[-(1 - \frac{1}{q^2})^{\nu-1/2}\right] - 1$. Then, $\gamma_\nu(q)$ is substituted into (16). The corresponding integral would have two parts as $C^{1,\nu}_{mn}$ and $C^{2,\nu}_{mn}$ which are shown in (18).

$$C^\nu_{mn} = \left[1 + (-1)^{m+n}\right] \int_{0}^{\infty} J_{n+\nu}(\varepsilon q) J_{m+\nu}(\varepsilon q) q^{2\nu-1} \gamma_\nu(q) dq = C^{1,\nu}_{mn} + C^{2,\nu}_{mn}$$  \hspace{1cm} (18)

Here,

$$C^{1,\nu}_{mn} = \left[1 + (-1)^{m+n}\right] \int_{0}^{\infty} J_{n+\nu}(\varepsilon q) J_{m+\nu}(\varepsilon q) \frac{dq}{q}$$

$$C^{2,\nu}_{mn} = \left[1 + (-1)^{m+n}\right] \int_{0}^{\infty} J_{n+\nu}(\varepsilon q) J_{m+\nu}(\varepsilon q) \gamma_\nu(q) \frac{dq}{q}.$$  \hspace{1cm} (18)
\( C_{mn}^{1,\nu} \) has an analytical solution \([14, 15]\). The result of the integral \( C_{mn}^{1,\nu} \) given in (19).

\[
C_{mn}^{1,\nu} = \left[ 1 + (-1)^{m+n} \right] \frac{2}{\pi(m+n+2\nu)(m-n)} \sin\left(\frac{m-n}{2}\pi\right)
\]  

(19)

where, \( Re(m+\nu, n+\nu) > 0 \).

On the other hand, the integral \( C_{mn}^{2,\nu} \) is taken numerically. When \( q \) goes to infinity \((q \to \infty)\), \( C_{mn}^{2,\nu} \) has the behavior as \( O\left(\frac{1}{q^{2(1+\nu)}}\right) \).

4. Physical Characteristics of the Electric Field

In this section, the total radar cross section (TRCS) and the relation between the impedance and the fractional order are given. After having the expression for \( F^{1-\nu} \) given as (13), the radiation pattern of the scattered electric field in the far zone can be found by using (20). For the large values of \( ka \) \((ka \to \infty)\), the scattered electric field gets the next form by the stationary phase method \([8]\). Here, \( x = r\cos \phi \) and \( y = r\sin \phi \).

\[
E_{s}^{x}(x, y) = A(kr)\Phi^{\nu}(\phi)
\]  

(20)

where \( A(kr) = \sqrt{\frac{2}{\pi kr}} e^{\frac{ikr-\frac{\pi}{4}}{2}} \) and \( \Phi^{\nu} = -\frac{1}{4}(\pm)^{\nu} F^{1-\nu}(\cos \phi) \sin^{\nu}(\phi) \).

In (20), \( \Phi^{\nu}(\phi) \) is denoted as Radiation Pattern (RP). The upper sign in \( \Phi^{\nu} \) expression is chosen for \( \phi \in [0, \pi] \), and the lower sign stands for \( \phi \in [\pi, 2\pi] \) where \( \phi \) is the observation angle. \( A(kr) \) corresponds to the radial and \( \Phi^{\nu}(\phi) \) is the angular part of the scattered electric field in the far zone.

Total radar cross section is another investigation of physical characteristics of the electric field. In order to find TRCS, (21) is taken into account \([16]\).

\[
\sigma_{t} = \int_{0}^{2\pi} |\Phi^{\nu}|^{2} \, d\phi
\]  

(21)

The other important physical characteristic of the strip is the impedance. The normalized impedance of the strip can be found as \( \eta_{\nu} = -\frac{1}{\sin \theta} \tan\left(\frac{\pi \nu}{2}\right) \)[7, 9]. Then, the fractional order for the specific impedance value of the surface can be found by using (22). Throughout this paper, the impedance, permittivity and permeability are assumed to be relative values with respect to the vacuum.

\[
\nu = \frac{1}{i\pi} \ln\left(\frac{1 - \eta_{\nu} \sin \theta}{1 + \eta_{\nu} \sin \theta}\right)
\]  

(22)

For the normal incidence case, (22) becomes \( \nu = \frac{1}{i\pi} \ln\left(\frac{1 - \eta_{\nu}}{1 + \eta_{\nu}}\right) \) and this yields \( \eta_{\nu} = -i \tan\left(\frac{\pi}{2} \nu\right) \). When the fractional order \( \nu \) is complex-valued as \( \nu = \nu_{1} + i\nu_{2} \) where \( \nu_{1} \) and \( \nu_{2} \) are real numbers and the incidence angle is the normal incidence, the real and imaginary parts of the impedance can be found as (23) and (24). Here,
the impedance can also be denoted as \( \eta_\nu = \eta_1 + i\eta_2 \) where \( \eta_1 \) and \( \eta_2 \) are real numbers.

\[
\begin{align*}
\eta_1 &= \frac{\tanh \left( \frac{\pi}{2} \nu_2 \right) \tan^2 \left( \frac{\pi}{2} \nu_1 \right) \tanh \left( \frac{\pi}{2} \nu_2 \right)}{1 + \tan^2 \left( \frac{\pi}{2} \nu_1 \right) \tanh^2 \left( \frac{\pi}{2} \nu_2 \right)} \\
\eta_2 &= -\frac{\tan \left( \frac{\pi}{2} \nu_1 \right) - \tan \left( \frac{\pi}{2} \nu_1 \right) \tan^2 \left( \frac{\pi}{2} \nu_2 \right)}{1 + \tan^2 \left( \frac{\pi}{2} \nu_1 \right) \tanh^2 \left( \frac{\pi}{2} \nu_2 \right)}
\end{align*}
\]

By the definition, the impedance can be expressed with the following expression \( \eta = \sqrt{\tau} \) where \( \mu \) stands for the magnetic permeability and \( \epsilon \), in general, is called the complex permittivity. The permittivity can be expressed as \( \epsilon = \epsilon' + i\epsilon'' \) where \( \epsilon' \) is the electric permittivity while \( \epsilon'' \) characterizes the absorption or the loss [17]. After some mathematical transformations by using 23 and 24, the real and the imaginary part of the complex permittivity can be found as follows

\[
\epsilon' = \mu \frac{\eta_1^2 - \eta_2^2}{(\eta_1^2 - \eta_2^2)^2 + 4\eta_1^2\eta_2^2} \tag{25}
\]

\[
\epsilon'' = -\mu \frac{2\eta_1\eta_2}{(\eta_1^2 - \eta_2^2)^2 + 4\eta_1^2\eta_2^2} \tag{26}
\]

5. Numerical Analysis and Results

In this part, Total Radar Cross Section (TRCS), the near electric field distribution, the impedance and the complex permittivity variation with respect to the fractional order are given. Also, the comparison of radiation patterns obtained by the fractional boundary condition and the impedance boundary condition is done. In the TRCS, the value of the Total Radar Cross Section is shown with respect to \( ka \). Note that, fractional order, \( \nu \) can be denoted as \( \nu = \nu_1 + i\nu_2 \) where \( \nu_1 \) and \( \nu_2 \) are real numbers. Also, the impedance can be denoted as \( \eta_\nu \) and the real and the imaginary parts of the impedance can be written as \( \eta_\nu = \eta_1 + i\eta_2 \) where \( \eta_1 \) and \( \eta_2 \) are real numbers. In Figure 2, there is a family of graphs in which TRCS of different fractional orders with normal incidence is investigated. The lateral straight line in red corresponds to \( \log_{10}(TRCS) = 0 \). Here, the half width of the strip \( (a) \) is equal to 1. The real part of fractional order, \( \nu_1 \) is 0.75, and the imaginary part of fractional order, \( \nu_2 \) changes from 0.1 to 0.8 with the steps 0.1. As it is seen in Figure 2, the resonance peaks are observed. By changing the imaginary part of fractional order, the sharpness of resonances changes. The resonance becomes sharper when the imaginary part, \( \nu_2 \) is increased up to 0.7.

While having the imaginary part of the fractional order as 0.6, TRCS is reaching beyond 1.2 on the logarithmic scale (\( \log_{10}(TRCS) \)), whereas, when the imaginary part of the fractional order is 0.5, TRCS becomes approximately 0.65 on the logarithm scale. For the imaginary part of fractional order which has the value from 0.1 to 0.4, the peak values change slowly, on the other hand, when the imaginary part of the fractional order is between 0.4 and 0.7, the peak values are increasing very rapidly. For the imaginary part of fractional order \( \nu_2 = 0.7 \), the peak value becomes 3.3 on the logarithmic scale. While the imaginary part of the fractional order, \( \nu_2 \) is equal to 0.8, again the peak value is decreased. The highest peak value is the case that \( \nu_2 \) is equal to 0.7.

Figure 3 corresponds to the real part of impedance \( \eta_1 \) when the real part of fractional order \( \nu_1 \) is equal to 0.75. In the figure, the imaginary part of fractional order, \( \nu_2 \) changes from 0.1 to 0.8. The impedance has
both real and imaginary parts. The maximum value for the real part of the impedance is observed when the imaginary part of fractional order, $\nu_2$ is equal to approximately 0.3. Figure 4 corresponds to the complex part of impedance $\eta_\nu$ when $\nu_1$ is equal to 0.75 and $\nu_2$ changes from 0 to 0.8. The complex part of impedance changes the sign when $\nu_2$ is equal to 0.5. Here, the lateral straight line corresponds to the axis where the imaginary part of impedance, $\eta_2$ is equal to zero. When $\nu_2 = 0.5$, the impedance has only the real part. In both Figure 3 and Figure 4, the incidence wave is the normal incidence and the real part of fractional order, $\nu_1$ has the value of 0.75 which is close to the perfect magnetic conductor case.

In Figure 5, there is a family of graphs in which TRCS of different fractional orders with normal incidence is investigated. Here, the half width of the strip ($a$) is again equal to 1. In this case, the real part of fractional order, $\nu_1$ is 0.2, and the imaginary part of fractional order, $\nu_2$ changes from 0.3 to 0.5 with the steps 0.1. As it is seen in Figure 5, the resonance peaks are observed. By changing the imaginary part of the fractional order, the sharpness of resonances changes. When the imaginary part of the fractional order is between 0.3 and 0.4, the peak values change slowly. On the other hand, the peak values are increasing very rapidly and the peaks have higher values when the imaginary part of the fractional order is between 0.4 and 0.6. As it is seen in Figure 5, the maximum peak value becomes 3.64 when the imaginary part of fractional order, $\nu_2$ is equal to 0.6. Then, the peak value decreases again when the imaginary part of fractional order, $\nu_2$ is higher than 0.6.

Figure 6 shows the relation between the real part of impedance $\eta_1$ and the imaginary part of fractional order, $\nu_2$ when the real part of fractional order, $\nu_1$ is equal to 0.2. In the figure, the imaginary part of fractional order, $\nu_2$ changes from 0.1 to 0.7. In this case, on the contrary to the previous case given in Figure 3, there is no local maximum. Figure 7 corresponds to the complex part of impedance $\eta_\nu$ when $\nu_1$ is equal to 0.2 and $\nu_2$ changes from 0.1 to 0.7. As we see, the complex part of the impedance changes the sign when $\nu_2$ is equal to 0.5. The Lateral straight red line corresponds to the axis where the imaginary part of the impedance $\eta_2$ is equal to zero. In both Figure 6 and Figure 7, the incidence wave is the normal incidence and the real part of
Figure 3. The dependency of impedance’s real part on the imaginary part of fractional order when the real part of fractional order, $\nu_1$ is equal to 0.75

Figure 4. The dependency of impedance’s imaginary part on the imaginary part of fractional order when the real part of fractional order, $\nu_1$ is equal to 0.75

Figure 5. TRCS for 1. $\nu = 0.2 + i0.3$, 2. $\nu = 0.2 + i0.4$, 3. $\nu = 0.2 + i0.5$, 4. $\nu = 0.2 + i0.6$, 5. $\nu = 0.2 + i0.7$, 6. $\nu = 0.2 + i0.8$, $a = 1$, $\theta = \pi/2$
fractional order, $\nu_1$ is 0.2 which is close to the perfect electric conductor.

In Figure 8, there is again a family of graphs in which the TRCS of different fractional orders with normal incidence is studied. Here, the half width of the strip (a) is equal to 1. The real part of fractional order, $\nu_1$ is equal to 0.5, and the imaginary part of fractional order, $\nu_2$ changes from 0.1 to 0.5 with steps 0.1. As it is seen in Figure 8, resonance peaks are observed. The values of resonances change with the imaginary part of the fractional order. The resonance becomes sharper when the imaginary part is increased up to 0.7. When the imaginary part is equal to 0.5, the TRCS is reaching beyond 0.72 on the logarithmic scale. TRCS becomes approximately 0.35 on the logarithmic scale when the imaginary part of the fractional order is 0.4. For the imaginary part of the fractional order which is between 0.1 to 0.4, the peak values change slowly, on the other hand, the peak values are increasing very rapidly in the case of having the imaginary part of the fractional order which has the values between 0.4 and 0.7. Note that, the imaginary part of fractional order, $\nu_2$ has a value of 0.8, again the peak value is decreased. Therefore, the highest peak value is the case that $\nu_2$ is equal to 0.7.

Figure 9 investigates the relation between the real part of impedance $\eta_1$ and the imaginary part of fractional order, $\nu_2$ when the real part of fractional order, $\nu_1$ is equal to 0.5. In this case, on the contrary to the previous case given in Figure 3, there is no local maximum. Figure 10 corresponds to the complex part of impedance $\eta_\nu$ when $\nu_1$ is 0.5. Here, $\nu_2$ changes from 0.1 to 0.8 and the complex part of the impedance changes the sign when $\nu_2$ is equal to 0.5. The lateral straight line in Figure 10 corresponds to the axis where the imaginary part of impedance, $\eta_2$ is equal to zero. When $\nu_2$ is equal to 0.5, the impedance has only the real part. In both Figure 9 and Figure 10, the incidence wave is the normal incidence and the real part of fractional order, $\nu_1$ is 0.5 which is exactly middle of the perfect electric conductor and the perfect magnetic conductor case.

Between Figure 11 and Figure 16, the distributions of the total electric field are given. As it can be seen in Figure 11, the scattered electric field dominates total electric field distribution which is symmetric with respect to y-axis where the strip is located. In Figure 2, Total Radar Cross Section gives the resonance $ka$ value of the strip with parameters given as $\nu = 0.75 + i0.7, \ a = 1, \ \theta = \pi/2$. As it can be seen in Figure 13, at the resonance case ($ka=0.495$) found in Figure 2, the near field distribution has a peak value of approximately 40. This is the maximum value of the E – field amplitude for given parameters. In Figure 12, the near field...
Figure 8. TRCS for 1. $\nu = 0.5 + i0.1$, 2. $\nu = 0.5 + i0.2$, 3. $\nu = 0.5 + i0.3$, 4. $\nu = 0.5 + i0.4$, 5. $\nu = 0.5 + i0.5$, 6. $\nu = 0.5 + i0.6$, 7. $\nu = 0.5 + i0.7$, 8. $\nu = 0.5 + i0.8$, $a = 1$, $\theta = \pi/2$

Figure 9. The dependency of impedance’s real part on the imaginary part of fractional order when the real part of fractional order, $\nu_1$ is equal to 0.5

Figure 10. The dependency of impedance’s imaginary part on the imaginary part of fractional order when the real part of fractional order, $\nu_1$ is equal to 0.5
distribution of total electric field is given for parameters listed as $k\alpha = 0.5$, $\nu = 0.75 + i 0.6$, $a = 1$ and $\theta = \pi/2$. In this case, the incidence wave dominates and the peak value is approximately reaching up to 4.25. When Figure 11 and Figure 12 are compared, it can be understood that the complex part of the fractional order effects the distribution of the E – Field in the region abruptly.

In Figure 13, again, the near field distribution of the total electric field is given. In the same figure, the scattered electric field dominates the incidence field. The field distribution is symmetric with respect to y-axis where the strip is located. In Figure 5, Total Radar Cross Section gives the resonance $k\alpha$ value of the strip with parameters as $\nu = 0.2 + i 0.6$, $a = 1$, $\theta = \pi/2$. As it can be seen in Figure 13, at the resonance case ($k\alpha=0.146$) found in Figure 5, the near field distribution has a peak value of 66. This is the maximum value of the E – field amplitude for given parameters. In Figure 14, the near field distribution of the total electric field is shown. Here the variables are listed as $k\alpha=0.169$, $\nu = 0.2 + i 0.7$, $a = 1$ and $\theta = \pi/2$. In this case, the incidence wave dominates and the peak value is approximately reaching up to 3.21. When Figure 13 and Figure 14 are compared, it can be understood that the complex part of the fractional order effects dramatically the distribution of the E – Field in the region.

In Figure 15, similarly, the near field distribution of the total electric field is illustrated. In the same figure, the scattered electric field dominates with respect to the incidence field. The field distribution is not symmetric completely with respect to y-axis where the strip is located because in this case, $k\alpha$ value is not the same with the resonance value but still, the scattering field dominates. As it can be seen in Figure 15, the near field distribution has a peak value of approximately 13.5 when $k\alpha$ chosen near the resonance value. In Figure 16, for parameters given as $k\alpha=0.331$, $\nu = 0.5 + i 0.6$, $a = 1$, and $\theta = \pi/2$, the near field distribution of the total electric field is given. In this case, the incidence wave dominates and the peak value is approximately reaching up to 4.51. When Figure 15 and Figure 16 are compared, again, it can be understood that the complex part of the fractional order plays an important role in the distribution of the E – Field in the region.

In Figure 17 and Figure 18, in order to verify the fractional derivative method (FDM), the comparisons of radiation pattern $\Phi^\nu(\phi)$ are done between the results obtained by the fractional boundary condition and impedance boundary condition (IBC) for two different values of the fractional order and corresponding impedance values ($\phi \in [0, \pi]$) [9]. For the fractional order $\nu = 0.5 + i 0.5$, the corresponding impedance value
Figure 13. The near field distribution of total electric field at the resonant frequency for $ka = 0.146, \nu = 0.2 + i0.6, a = 1, \theta = \frac{\pi}{2}$

Figure 14. The near field distribution of total electric field at the resonant frequency for $ka = 0.169, \nu = 0.2 + i0.7, a = 1, \theta = \frac{\pi}{2}$

Figure 15. The near field distribution of total electric field for $ka = 0.331, \nu = 0.5 + i0.7, a = 1, \theta = \frac{\pi}{2}$

Figure 16. The near field distribution of total electric field near for $ka = 0.331, \nu = 0.5 + i0.6, a = 1, \theta = \frac{\pi}{2}$
becomes $\eta = 0.9172$ and for the fractional order $\nu = 0.75 + i0.7$, the corresponding impedance value becomes $\eta = 1.1546 + i1.4546$ found by (23) and (24).

Figure 19 shows the graph on which the real part of the complex permittivity $\epsilon'$ changes when the imaginary part of the complex fractional order changes in the range $0.1 < \nu_2 < 0.8$ while $\nu_1$ is 0.75. There are the maximum and the minimum values in the range. Figure 20 shows the relation between the imaginary part of the complex permittivity $\epsilon''$ on the imaginary part of the complex fractional order when it changes in the range $0.1 < \nu_2 < 0.8$. When $\nu_2$ is 0.6, the graph has a minimum. Beside, in the range $0.5 < \nu_2 < 0.8$, $\epsilon''$ is negative-valued. This means that the strip with corresponding parameters has “negative absorption” [17]. This fact explains why there are peak values on Figure 2. As it can be seen in Figure 2, after the complex part of the fractional order $\nu_2$ reaching 0.5, TRCS values are increasing sharply for the specific $ka$ values because the complex part of the permittivity becomes negative for this range as given in Figure 20. When the complex part of the fractional order $\nu_2$ becomes 0.7, TRCS has the maximum value. After that, the resonance values are decreasing for increasing values of the complex part of the fractional order $\nu_2$ because, for this case, also the complex part of the permittivity is again closing the zero value. In other words, when the complex part of the permittivity $\epsilon''$ is getting negative minimum values, the resonance values increasing because the negative absorption property of the material is increasing as seen in Figure 20.

6. Conclusion
In this article, it was considered the plane electromagnetic wave diffraction by the strip. The two-dimensional problem is solved when the strip is infinite in $z$-direction. As the boundary condition, the fractional boundary condition with complex-valued fractional order is required on the surface of the strip. The complex-valued fractional order gave interesting results. In the results, the new type of resonance is observed, and its property was studied using TRCS and the near field distribution. This resonance does not exist for real fractional order. Therefore this is a unique property of the complex-valued fractional order boundary condition. The corresponding impedance value of the strip was also demonstrated.

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