The Meyer function on the handlebody group

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Abstract: We give an explicit formula for the signature of handlebody bundles over the circle in terms of the homological monodromy. This gives a cobounding function of Meyer’s signature cocycle on the mapping class group of a 3-dimensional handlebody, i.e., the handlebody group. As an application, we give a topological interpretation for the generator of the first cohomology group of the hyperelliptic handlebody group.

Key words: Signature cocycle, Handlebody group, Mapping class groups

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1. Introduction

Let $\Sigma_g$ be a closed connected oriented surface of genus $g$ and $\text{Mod}(\Sigma_g)$ the mapping class group of $\Sigma_g$, namely the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_g$. Unless otherwise stated, we assume that (co)homology groups have coefficients in $\mathbb{Z}$. The second cohomology of $\text{Mod}(\Sigma_g)$ has been determined for all $g \geq 1$ by works of many people, in particular by the seminal work of Harer [6, 7] for $g \geq 3$. We have $H^2(\text{Mod}(\Sigma_1)) \cong \mathbb{Z}/12\mathbb{Z}$, $H^2(\text{Mod}(\Sigma_2)) \cong \mathbb{Z}/10\mathbb{Z}$, and

$$H^2(\text{Mod}(\Sigma_g)) \cong \mathbb{Z} \quad \text{for } g \geq 3.$$

There are various interesting constructions of non-trivial second cohomology class of $\text{Mod}(\Sigma_g)$; the reader is referred to the survey article [13]. Among others, the remarkable approach of Meyer [16, 17] was to consider the signature of $\Sigma_g$-bundles over surfaces. The central object that Meyer used was a normalized 2-cocycle

$$\tau_g : \text{Sp}(2g; \mathbb{Z}) \times \text{Sp}(2g; \mathbb{Z}) \to \mathbb{Z}$$

on the integral symplectic group of degree $2g$.

Meyer showed that for $g \geq 3$ the pullback of the cohomology class of $\tau_g$ by the homology representation $\rho : \text{Mod}(\Sigma_g) \to \text{Sp}(2g; \mathbb{Z})$ is of infinite order in $H^2(\text{Mod}(\Sigma_g))$. On the other hand, if $g = 1, 2$ then $[\rho^* \tau_g]$ is torsion and there exists a (unique) rational valued cobounding function $\phi_g : \text{Mod}(\Sigma_g) \to \mathbb{Q}$ of $\rho^* \tau_g$. This means that

$$\tau_g(\rho(\varphi_1), \rho(\varphi_2)) = \phi_g(\varphi_1) + \phi_g(\varphi_2) - \phi_g(\varphi_1 \varphi_2) \quad \text{for any } \varphi_1, \varphi_2 \in \text{Mod}(\Sigma_g).$$

Since the case $g = 1$ was extensively studied by Meyer, such a cobounding function is called a Meyer function. Some number-theoretic and differential geometric aspects of the function $\phi_1$
were studied by Atiyah [2]. The case $g = 2$ was studied by Matsumoto [15], Morifuji [18] and Iida [11]. For $g \geq 3$, there is no cobounding function of $\rho^*\tau_g$ on the whole mapping class group $\text{Mod}(\Sigma_g)$. However, if we restrict $\rho^*\tau_g$ to a subgroup called the hyperelliptic mapping class group $\mathcal{H}(\Sigma_g)$, then it is known that there is a (unique) cobounding function $\phi^H_g : \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ of $\rho^*\tau_g$. Note that $\mathcal{H}(\Sigma_g) = \text{Mod}(\Sigma_g)$ for $g = 1, 2$. This function $\phi^H_g$ was studied by Endo [4] and Morifuji [18]. One motivation for studying Meyer functions comes from the localization phenomenon of the signature of fibered 4-manifolds. See, e.g., [1, 14].

In this paper, we study a new example of Meyer functions: the Meyer function on the handlebody group. The handlebody group of genus $g$, which we denote by $\text{Mod}(V_g)$, is defined as the group of isotopy classes of orientation-preserving self-diffeomorphisms of the 3-dimensional handlebody $V_g$ of genus $g$. It is well known that the natural homomorphism $\text{Mod}(V_g) \to \text{Mod}(\Sigma_g)$, $\varphi \mapsto \varphi|_{\Sigma_g}$ is injective since $V_g$ is an irreducible 3-manifold. Therefore, we can think of $\text{Mod}(V_g)$ as a subgroup of $\text{Mod}(\Sigma_g)$. For a mapping class $\varphi \in \text{Mod}(V_g)$, we denote by $M_\varphi$ the mapping torus of $\varphi$. It is a compact oriented 4-manifold. We define

$$
\phi^V_g(\varphi) := \text{Sign } M_\varphi \in \mathbb{Z}.
$$

We show in Lemma 4.2 that $\phi^V_g$ is a cobounding function of the cocycle $\rho^*\tau_g$ on the handlebody group $\text{Mod}(V_g)$. If $g \geq 3$, this is the unique cobounding function since $H_1(\text{Mod}(V_g))$ is torsion (see [21, Theorem 20] and [12, Remark 3.5]).

The value $\phi^V_g(\varphi)$ can be computed from the action of $\varphi$ on the first homology $H_1(\Sigma_g)$, and our first result gives its explicit description. To state it, we take a suitable basis of $H_1(\Sigma_g)$ so that the homology representation $\rho$ restricted to $\text{Mod}(V_g)$ takes values in a subgroup $\text{urSp}(2g; \mathbb{Z}) \subset \text{Sp}(2g; \mathbb{Z})$. (See Section 2.3 for details.) Then, $\rho(\varphi)$ is of the form $\rho(\varphi) = \left(\begin{array}{cc} P & Q \\ O_g & S \end{array}\right)$, where $P$, $Q$ and $S$ are $g \times g$ matrices. We consider a $\mathbb{Q}$-linear space $U_\varphi := \ldots$
Ker\((S - I_g) \subset \mathbb{Q}^g\), and define a bilinear form \(\langle , \rangle_\varphi\) on it by
\[
\langle x, y \rangle_\varphi := \langle x^t Q y \rangle, \text{ for } x, y \in U_\varphi.
\]
It turns out that \(\langle , \rangle_\varphi\) is symmetric, and we have the following:

**Theorem 1.1** The value \(\phi_g^V(\varphi)\) coincides with the signature of the symmetric bilinear form \(\langle , \rangle_\varphi\) on \(U_\varphi\).

In fact, we will show in Section 3.5 that the intersection form on \(H_2(M_\varphi)\) is equivalent to the bilinear form \(\langle , \rangle_\varphi\).

As a corollary, we see that the function \(\phi_g^V\) is bounded by \(g = \text{rank} H_1(V_g)\). We also give sample calculations of \(\phi_g^V\) in Lemmas 4.4 and 4.5. Walker also constructed a function \(j : \text{Mod}(\Sigma_g) \to \mathbb{Q}\) whose restriction to \(\text{Mod}(V_g)\) coincides with \(\phi_g^V\). Our description of \(\phi_g^V\) in Theorem 1.1 is similar to but different from a description of \(j\) given by Gilmer and Masbaum [5, Proposition 6.9]. See, for details, Remark 3.6.

As an application of the function \(\phi_g^V\), we obtain a non-trivial first cohomology class in the intersection \(\mathcal{H}(\Sigma_g) \cap \text{Mod}(V_g)\) called the hyperelliptic handlebody group, denoted by \(\mathcal{H}(V_g)\). The group \(\mathcal{H}(V_g)\) is an extension by \(\mathbb{Z}/2\mathbb{Z}\) of a subgroup of the mapping class group of a 2-sphere with \((2g + 2)\)-punctures, called the Hilden group. The Hilden group was introduced in [8], and it is related to the study of links in 3-manifolds. In [10], Hirose and Kin studied the minimal dilatation of pseudo-Anosov elements in \(\mathcal{H}(V_g)\), and gave a presentation of \(\mathcal{H}(V_g)\).

We consider the difference
\[
\phi_g^\mathcal{H} - \phi_g^V \in \text{Hom}(\mathcal{H}(V_g), \mathbb{Q}) = H^1(\mathcal{H}(V_g); \mathbb{Q})
\]
of the Meyer functions on \(\mathcal{H}(\Sigma_g)\) and on \(\text{Mod}(V_g)\). From the abelianization of \(\mathcal{H}(V_g)\) obtained in [10, Corollary A.9], we see that the rank of \(H^1(\mathcal{H}(V_g))\) is one. Let us denote a generator of
Our second result is:

**Theorem 1.2** Let \( g \geq 1 \). We have

\[
\phi^H_g - \phi^V_g = \begin{cases} 
\frac{2}{2g+1} \mu & \text{if } g \text{ is even}, \\
\frac{1}{2g+1} \mu & \text{if } g \text{ is odd}.
\end{cases}
\]

When \( g = 1, 2 \), we have \( \mathcal{H}(V_g) = \text{Mod}(V_g) \), and \( \phi^H_g - \phi^V_g \) gives an abelian quotient of \( \text{Mod}(V_g) \).

There is an interpretation of the cohomology class \( \phi^H_g - \phi^V_g \) in terms of a kind of connecting homomorphism. We assume that \( g \geq 3 \). From the diagram

\[
\begin{array}{ccc}
\mathcal{H}(V_g) & \xrightarrow{i_2} & \text{Mod}(V_g) \\
\downarrow{i_1} & & \downarrow{j_2} \\
\mathcal{H}(\Sigma_g) & \xrightarrow{j_1} & \text{Mod}(\Sigma_g). \\
\end{array}
\]

of groups and their inclusions, we have a natural homomorphism

\[
\Upsilon: H^2(\text{Mod}(\Sigma_g); \mathbb{Q}) \to H^1(\mathcal{H}(V_g); \mathbb{Q})
\]

defined as follows. For \([c] \in H^2(\text{Mod}(\Sigma_g); \mathbb{Q})\), there are cobounding functions \( f^H: \mathcal{H}(\Sigma_g) \to \mathbb{Q} \) of \( j_1^*c \) and \( f^V: \text{Mod}(V_g) \to \mathbb{Q} \) of \( j_2^*c \), respectively. The cochain \( i_1^*f^H - i_2^*f^V \) is actually a homomorphism on \( \mathcal{H}(V_g) \). It does not depend on the choices of the representatives \( c, f^H, \) and \( f^V \) since \( H^1(\text{Mod}(V_g); \mathbb{Q}) = H^1(\mathcal{H}(\Sigma_g); \mathbb{Q}) = 0 \) when \( g \geq 3 \). Then \( \Upsilon([c]) \) is defined to be \( i_1^*f^H - i_2^*f^V \). In this setting, our cohomology class is written as \( \Upsilon([\tau_g]) = \phi^H_g - \phi^V_g \in H^1(\mathcal{H}(V_g); \mathbb{Q}) \).

The outline of this paper is as follows. In Section 2, we review the definition of Meyer’s signature cocycle and the handlebody group \( \text{Mod}(V_g) \). We also review the abelianization of the
hyperelliptic handlebody group obtained in [10], and describe a generator of the cohomology
group $H^1(\mathcal{H}(V_g))$ in Corollary 2.6. In Section 3, we investigate the intersection form of the map-
ting torus of $\varphi \in \text{Mod}(V_g)$, and prove Theorem 1.1. As it turns out, we can explicitly describe
$\phi_g^V$ as a function on a subgroup $\text{urSp}(2g; \mathbb{Z})$ of the integral symplectic group. In Section 4, we
prove Theorem 1.2 by using explicit calculations of the Meyer function $\phi_g^V : \text{Mod}(V_g) \to \mathbb{Z}$ in
Lemmas 4.4 and 4.5.

2. Preliminaries on mapping class groups

Fix a non-negative integer $g$.

2.1. Mapping class group of a surface

Let $\Sigma_g$ be a closed connected oriented surface of genus $g$. The mapping class group of $\Sigma_g$, de-
noted by $\text{Mod}(\Sigma_g)$, is the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_g$. To simplify notation, we will use the same letter for a self-diffeomorphism of $\Sigma_g$ and its
isotopy class.

The first homology group $H_1(\Sigma_g)$ is equipped with a non-degenerate skew-symmetric pairing $\langle \cdot, \cdot \rangle$, namely the intersection form. Thus we can take a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$
for $H_1(\Sigma_g)$. This means that $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$ and $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$ for any $i, j \in \{1, \ldots, g\}$,
where $\delta_{ij}$ is the Kronecker symbol.

Once a symplectic basis for $H_1(\Sigma_g)$ is fixed, we obtain the homology representation

$$\rho : \text{Mod}(\Sigma_g) \to \text{Sp}(2g; \mathbb{Z}), \quad \varphi \mapsto \varphi_*.$$  

Here, the target is the integral symplectic group

$$\text{Sp}(2g; \mathbb{Z}) = \{ A \in \text{GL}(2g; \mathbb{Z}) \mid {}^tAJA = J \},$$
where \( J = \begin{pmatrix} O_g & I_g \\ -I_g & O_g \end{pmatrix} \), and \( \rho(\varphi) = \varphi_* \) is the matrix presentation of the action of \( \varphi \) on \( H_1(\Sigma_g) \) with respect to the fixed symplectic basis. We use block matrices to denote elements in \( \text{Sp}(2g;\mathbb{Z}) \), e.g., \( A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \) with \( g \times g \) integral matrices \( P, Q, R, \) and \( S \).

### 2.2. Meyer’s signature cocycle

Let \( A, B \in \text{Sp}(2g;\mathbb{Z}) \). We consider an \( \mathbb{R} \)-linear space

\[
V_{A,B} := \{ (x, y) \in \mathbb{R}^{2g} \oplus \mathbb{R}^{2g} \mid (A^{-1} - I_{2g})x + (B - I_{2g})y = 0 \}
\]

and a bilinear form on \( V_{A,B} \) given by

\[
\langle (x, y), (x', y') \rangle_{A,B} := (x + y)J(I_{2g} - B)y'.
\]

The form \( \langle \cdot, \cdot \rangle_{A,B} \) turns out to be symmetric, and thus its signature is defined; we set

\[
\tau_g(A, B) := \text{Sign}(V_{A,B}, \langle \cdot, \cdot \rangle_{A,B}).
\]

The map \( \tau_g : \text{Sp}(2g;\mathbb{Z}) \times \text{Sp}(2g;\mathbb{Z}) \rightarrow \mathbb{Z} \) is called Meyer’s signature cocycle [16, 17]. It is a normalized 2-cocycle of the group \( \text{Sp}(2g;\mathbb{Z}) \).

Let \( P \) be a compact oriented surface of genus 0 with three boundary components, i.e., a pair of pants. We denote by \( C_1, C_2 \) and \( C_3 \) the boundary components of \( P \). Choose a base point in \( P \), and let \( \ell_1, \ell_2 \) and \( \ell_3 \) be based loops in \( P \) such that \( \ell_i \) is parallel to the negatively oriented boundary component \( C_i \) for any \( i \in \{1, 2, 3\} \) and \( \ell_1\ell_2\ell_3 = 1 \) holds in the fundamental group \( \pi_1(P) \).

For given two mapping classes \( \varphi_1, \varphi_2 \in \text{Mod}(\Sigma_g) \), there is an oriented \( \Sigma_g \)-bundle
$E(\varphi_1, \varphi_2) \to P$ such that the monodromy along $\ell_i$ is $\varphi_i$ for $i = 1, 2$. It is unique up to bundle isomorphisms. The total space $E(\varphi_1, \varphi_2)$ is a compact 4-manifold equipped with a natural orientation, and hence its signature is defined.

**Proposition 2.1** (Meyer [16, 17]) $\text{Sign}(E(\varphi_1, \varphi_2)) = \tau_g(\rho(\varphi_1), \rho(\varphi_2))$.

**Remark 2.2** Turaev [20] independently found the signature cocycle. He also studied its relation to the Maslov index.

### 2.3. Handlebody group

Let $V_g$ be a handlebody of genus $g$. That is, $V_g$ is obtained by attaching $g$ one-handles to the 3-ball $D^3$. We identify $\Sigma_g$ and the boundary of $V_g$ by choosing an orientation-preserving diffeomorphism between them. We have the following short exact sequence

$$0 \to H_2(V_g, \Sigma_g) \xrightarrow{\partial_*} H_1(\Sigma_g) \xrightarrow{i_*} H_1(V_g) \to 0 \quad (2.1)$$

which is a part of the homology exact sequence of the pair $(V_g, \Sigma_g)$. There are properly embedded, oriented and pairwise disjoint disks $D_1, \ldots, D_g$ in $V_g$ whose homology classes (denoted by the same letters) constitute a basis for $H_2(V_g, \Sigma_g)$. We set $\alpha_i := \partial_*(D_i) \in H_1(\Sigma_g)$ for $i \in \{1, \ldots, g\}$. Then $\alpha_i$’s extend to a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for $H_1(\Sigma_g)$.

In what follows, we fix a symplectic basis obtained in this way. The image of the homology classes $\beta_1, \ldots, \beta_g$ by the map $i_*$ constitute a basis for $H_1(V_g)$. For simplicity, we denote them by the same letters $\beta_1, \ldots, \beta_g$.

We denote by $\text{Mod}(V_g)$ the *handlebody group* of genus $g$. It can be considered as a subgroup of $\text{Mod}(\Sigma_g)$. For any $\varphi \in \text{Mod}(V_g)$, the matrix $\rho(\varphi)$ lies in the subgroup of $\text{Sp}(2g; \mathbb{Z})$ defined by

$$\text{urSp}(2g; \mathbb{Z}) := \left\{ A \in \text{Sp}(2g; \mathbb{Z}) \mid A = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \right\},$$

for
cf. [3, 9] for details. The matrices \( P \), \( Q \) and \( S \) satisfy the following relations:

\[
\iota PS = I_g, \quad \iota QS = \iota SQ.
\]

(2.2)

Remark 2.3 The group \( \text{Mod}(V_g) \) acts naturally on the groups in (2.1), and the maps \( \partial_* \) and \( \iota_* \) are \( \text{Mod}(V_g) \)-module homomorphisms. The matrix presentation of the action \( \varphi_* \) on \( H_1(V_g) \) is \( S \).

2.4. Hyperelliptic handlebody group

An involution of \( \Sigma_g \) is called hyperelliptic if it acts on \( H_1(\Sigma_g) \) as \(-\text{id}\). We fix an hyperelliptic involution \( \iota \) which extends to an involution of \( V_g \), as in Figure.

\[\begin{array}{c}
\text{Figure.} \text{ The involution } \iota \text{ of } V_g \text{ and the curves } C_1, C_2, C_3.
\end{array}\]

The hyperelliptic mapping class group \( \mathcal{H}(\Sigma_g) \) is the centralizer of \( \iota \) in \( \text{Mod}(\Sigma_g) \):

\[
\mathcal{H}(\Sigma_g) := \{ \varphi \in \text{Mod}(\Sigma_g) \mid \varphi \iota = \iota \varphi \}.
\]

Definition 2.4 ([10]) The hyperelliptic handlebody group \( \mathcal{H}(V_g) \) is defined by

\[
\mathcal{H}(V_g) := \mathcal{H}(\Sigma_g) \cap \text{Mod}(V_g).
\]

Hirose and Kin [10, Appendix A] gave a finite presentation of the group \( \mathcal{H}(V_g) \). Moreover
they determined the abelianization of $\mathcal{H}(V_g)$ as

$$\mathcal{H}(V_g)^{\text{abel}} \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

for $g \geq 2$.

In fact, using their presentation, it is easy to make this result more explicit. Let $C_1, C_2$ and $C_3$ be simple closed curves on $\Sigma_g$ as in Figure. For each $i \in \{1, 2, 3\}$ denote by $t_i$ the right handed Dehn twist along $C_i$. Following [10], set $r_1 = t_2^{-1}t_3^{-1}t_1t_2$ and $s_1 = t_2t_3t_1t_2$. (Note that in [10], $t_C$ denotes the left handed Dehn twist along $C$.)

**Lemma 2.5** When $g = 1$, one has $\mathcal{H}(V_1) \cong \mathbb{Z} [t_1s_1] \oplus \mathbb{Z}_2 [t_1^2s_1]$. If $g \geq 2$, then

$$\mathcal{H}(V_g)^{\text{abel}} \cong \begin{cases} 
\mathbb{Z} [s_1] \oplus \mathbb{Z}_2 [t_1s_1^g] \oplus \mathbb{Z}_2 [r_1] & \text{if } g \text{ is even}, \\
\mathbb{Z} [t_1s_1^{\frac{g+1}{2}}] \oplus \mathbb{Z}_2 [t_1^2s_1^g] \oplus \mathbb{Z}_2 [r_1] & \text{if } g \text{ is odd}.
\end{cases}$$

Here, $[s_1]$ is the class of $s_1$ in $\mathcal{H}(V_g)^{\text{abel}}$, and $\mathbb{Z} [s_1]$ is the infinite cyclic group generated by $[s_1]$, etc.

**Proof** The case $g = 1$ follows from the fact that $\mathcal{H}(V_1) \cong \text{Mod}(V_1)$ and a result of Wajnryb [21, Theorem 14]. Assume that $g \geq 2$. Using [10, Theorem A.8], one sees that $\mathcal{H}(V_g)^{\text{abel}}$ is generated by $[r_1], [s_1]$ and $[t_1]$ with the relations

$$2[r_1] = 0, \quad 4[t_1] + 2g [s_1] = 0, \quad 2(g + 1)[t_1] + g(g + 1)[s_1] = 0.$$ 

The assertion follows from these relations by a direct computation. □

The following corollary to Lemma 2.5 will be used in Section 4.4 to prove Theorem 1.2.

**Corollary 2.6** Let $g \geq 1$. There is a unique homomorphism $\mu : \mathcal{H}(V_g) \to \mathbb{Z}$ satisfying the following property:
(1) If $g$ is even, $\mu(s_1) = 1$ and $\mu(t_1) = -g/2$;

(2) If $g$ is odd, $\mu(t_1) = -g$, $\mu(s_1) = 2$, and thus $\mu(t_1s_1^{\frac{g+1}{2}}) = 1$.

Moreover, the first cohomology group $H^1(\mathcal{H}(V_g)) = \text{Hom}(\mathcal{H}(V_g), \mathbb{Z})$ is an infinite cyclic group generated by $\mu$.

3. Handlebody bundles over $S^1$

3.1. Mapping torus

Let $I = [0, 1]$ be the unit interval. By identifying the endpoints of $I$, we obtain the circle $S^1 = [0, 1]/0 \sim 1$. Let $\ell: I \to S^1$ be the natural projection. For $t \in I$, we set $[t] := \ell(t)$. Choose $[0]$ as a base point of $S^1$. Then the fundamental group $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of $\ell$.

In what follows, we use the following cell decomposition of $S^1$: the 0-cell is $e^0 = [0]$ and the 1-cell is $e^1 = S^1 \setminus e^0$. The map $\ell$ induces an orientation of $e^1$.

Let $\varphi \in \text{Mod}(V_g)$. The mapping torus of $\varphi$ is the quotient space

$$M_\varphi := (I \times V_g)/(0, x) \sim (1, \varphi(x)).$$

For $(t, x) \in I \times V_g$, its class in $M_\varphi$ is denoted by $[t, x]$. The natural projection $\pi: M_\varphi \to S^1$, $[t, x] \mapsto [t]$ is an oriented $V_g$-bundle, and the total space $M_\varphi$ is a compact 4-manifold with boundary equipped with a natural orientation. The pullback of $M_\varphi \to S^1$ by $\ell$ is a trivial $V_g$-bundle over $I$, and its trivialization is given by the map

$$\Phi: I \times V_g \to M_\varphi, \quad (t, x) \mapsto [t, x]. \quad (3.1)$$
The following composition of maps coincides with $\varphi$:

$$V_g^{0 \times \text{id}} \cong \{0\} \times V_g \xrightarrow{\Phi(0, \cdot)} \pi^{-1}([0]) = \pi^{-1}([1]) \xrightarrow{\Phi(1, \cdot)^{-1}} \{1\} \times V_g^{1 \times \text{id}} \cong V_g.$$  

Therefore, the monodromy of $M_\varphi! : S^1 \to S^1$ along $\ell$ is equal to the mapping class $\varphi$. As was mentioned in Remark 2.3, the groups $H_2(V_g, \Sigma_g)$, $H_1(\Sigma_g)$, and $H_1(V_g)$ are $\text{Mod}(V_g)$-modules. Thus, these groups become $\pi_1(S^1)$-modules; the homotopy class of $\ell$, which is a generator of $\pi_1(S^1)$, acts as the monodromy $\varphi \in \text{Mod}(V_g)$.

### 3.2. Second homology of the mapping torus

For a non-negative integer $q \geq 0$, let $\mathcal{H}_q(V_g)$ be the local system on $S^1$ which comes from the $V_g$-bundle $\pi : M_\varphi \to S^1$, and whose fiber at $x \in S^1$ is the $q$-th homology group $H_q(\pi^{-1}(x))$. Similarly, we consider the local system $\mathcal{H}_q(V_g, \Sigma_g)$ whose fiber at $x \in S^1$ is the $q$-th relative homology group $H_q(\pi^{-1}(x), \partial \pi^{-1}(x))$.

Consider the Serre homology spectral sequence of the $V_g$-bundle $M_\varphi \to S^1$. It degenerates at the $E^2$ page, which is given by $E^2_{p,q} = H_p(S^1; \mathcal{H}_q(V_g))$. Since $H_2(V_g) = 0$ and the base space $S^1$ is 1-dimensional, we obtain

$$H_2(M_\varphi) \cong E^\infty_{1,1} \cong E^2_{1,1} = H_1(S^1; \mathcal{H}_1(V_g)).$$

Moreover, using the cellular homology of $S^1$ with coefficients in $\mathcal{H}_1(V_g)$, we have

$$H_1(S^1; \mathcal{H}_1(V_g)) \cong \ker(\partial : C_1(S^1; \mathcal{H}_1(V_g)) \to C_0(S^1; \mathcal{H}_1(V_g)))$$

$$= \ker(\partial : \mathbb{Z} e^1 \otimes H_1(V_g) \to \mathbb{Z} e^0 \otimes H_1(V_g) = H_1(V_g)),$$
where the boundary map is given by
\[
\partial(e^1 \otimes \alpha) = \ell_*(\alpha) - \alpha = (\Phi(0, \cdot)^{-1} \circ \Phi(1, \cdot))_*(\alpha) - \alpha = \varphi_*^{-1}(\alpha) - \alpha.
\]

In summary, we have proved the following lemma. In the statement, \(H_1(V_g)_{\pi_1(S^1)}\) is the space of invariants under the action of \(\pi_1(S^1)\), i.e., \(H_1(V_g)_{\pi_1(S^1)} = \{ \alpha \in H_1(V_g) \mid \varphi_*(\alpha) = \alpha \} \).

**Lemma 3.1** We have \(H_2(M_\varphi) \cong H_1(S^1; \mathcal{H}_1(V_g)) \cong H_1(V_g)_{\pi_1(S^1)}\).

Similarly, for the relative homology of the pair \((M_\varphi, \partial M_\varphi)\), there is a spectral sequence converging to \(H_*(M_\varphi, \partial M_\varphi)\) such that \(E^2_{p,q} = H_p(S^1; \mathcal{H}_q(V_g, \Sigma_g))\). This degenerates at the \(E^2\) page, too. Since \(H_1(V_g, \Sigma_g) = 0\), we obtain
\[
H_2(M_\varphi, \partial M_\varphi) \cong E^{\infty}_{0,2} \cong E^2_{0,2} = H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g)).
\]

By the same argument as above, we obtain the following lemma. In the statement, \(H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\) is the space of coinvariants under the action of \(\pi_1(S^1)\), i.e., the quotient of \(H_2(V_g, \Sigma_g)\) by the subgroup generated by the set \(\{ \varphi_*(\delta) - \delta \mid \delta \in H_2(V_g, \Sigma_g) \}\).

**Lemma 3.2** We have \(H_2(M_\varphi, \partial M_\varphi) \cong H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g)) \cong H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\).

**3.3. Description of the inclusion homomorphism**

Recall that the short exact sequence (2.1) is \(\text{Mod}(V_g)\)-equivariant. Let \(\alpha \in H_1(V_g)_{\pi_1(S^1)}\) be a \(\varphi_*\)-invariant homology class. Pick an element \(\tilde{\alpha} \in H_1(\Sigma_g)\) such that \(i_*(\tilde{\alpha}) = \alpha\). Then \(\varphi_*(\tilde{\alpha}) - \tilde{\alpha} \in \text{Ker}(i_*) = \text{Im}(\partial_*)\).

**Definition 3.3** \(d(\alpha) := [\partial_*^{-1}(\varphi_*(\tilde{\alpha}) - \tilde{\alpha})] \in H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\).

It is easy to see that \(d(\alpha)\) is independent of the choice of \(\tilde{\alpha}\). Thus we obtain a well-defined map \(d: H_1(V_g)_{\pi_1(S^1)} \to H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\).
Proposition 3.4 The following diagram is commutative:

\[
\begin{array}{ccc}
H_1(V_g)_{\pi_1(S^1)} & \xrightarrow{d} & H_2(V_g, \Sigma_g)_{\pi_1(S^1)} \\
\cong & & \cong \\
H_2(M_\varphi) & \xrightarrow{i_*} & H_2(M_\varphi, \partial M_\varphi),
\end{array}
\]

where the bottom horizontal arrow is the inclusion homomorphism, and the vertical arrows are the isomorphisms in Lemmas 3.1 and 3.2.

3.4. Proof of Proposition 3.4

In this section, for a topological space \(X\), we denote by \(S_n(X)\) and \(Z_n(X)\) the groups of singular \(n\)-chains and singular \(n\)-cycles, respectively.

Let \(\alpha \in H_1(V_g)_{\pi_1(S^1)}\). Pick its lift \(\tilde{\alpha} \in H_1(\Sigma_g)\) such that \(i_*(\tilde{\alpha}) = \alpha\). Take a singular 1-cycle \(\tilde{a} \in Z_1(\Sigma_g)\) representing the homology class \(\tilde{\alpha}\). Then, \(\varphi_\varphi^{-1}(\tilde{a}) - \tilde{a}\) is a singular 1-boundary in \(V_g\) since \(\varphi_\varphi^{-1}(\tilde{a}) - \tilde{a} \in \text{Ker}(i_*)\). Therefore, there exists \(\sigma_{\varphi, \alpha} \in S_2(V_g)\) such that

\[
\partial \sigma_{\varphi, \alpha} = \varphi_\varphi^{-1}(\tilde{a}) - \tilde{a}.
\]

First we compute the composition of \(d\) and the right vertical map. We claim that \(d(\alpha)\) is represented by the relative 2-cycle \(-\sigma_{\varphi, \alpha} \in Z_2(V_g, \Sigma_g)\). This follows from the equality \(\varphi_\varphi(\tilde{\alpha}) - \tilde{\alpha} = -(\varphi_\varphi^{-1}(\tilde{a}) - \tilde{a})\) in \(H_1(\Sigma_g)_{\pi_1(S^1)}\) and the relation \(\partial \sigma_{\varphi, \alpha} = \varphi_\varphi^{-1}(\tilde{a}) - \tilde{a}\). Hence, the right vertical map sends \(d(\alpha)\) to the homology class represented by the relative 2-cycle \(-e^0 \times \sigma_{\varphi, \alpha} \in Z_2(M_\varphi, \partial M_\varphi)\), where the symbol \(\times\) means the cross product.

Next we compute the composition of the left vertical map and \(i_*\). For this purpose, we set

\[
Z_\alpha := \Phi_\varphi(I \times \tilde{\alpha}) - e^0 \times \sigma_{\varphi, \alpha} \in S_2(M_\varphi).
\]

Here, \(\Phi\) is the map defined in (3.1), and the unit interval is regarded as a singular 1-chain in the obvious way. Actually, \(Z_\alpha\) is a 2-cycle in \(M_\varphi\).
Lemma 3.5 The isomorphism in Lemma 3.1 sends $\alpha$ to the homology class of $Z_\alpha$.

Proof We need to inspect the spectral sequence involved in Lemma 3.1. For simplicity we denote $M = M_\varphi$, and for every non-negative integer $q \geq 0$ let $M^{(q)}$ be the inverse image of the $q$-skeleton of $S^1$ by the projection map $\pi$. Thus we have $\emptyset \subset M^{(0)} = \pi^{-1}([0]) \subset M^{(1)} = M$.

Accordingly, the singular chain complex $S_*(M)$ has an increasing filtration: $\{0\} \subset S_*(M^{(0)}) \subset S_*(M^{(1)}) = S_*(M)$. The associated spectral sequence is the one that we consider.

Now let $\alpha \in H_1(V_g)^{\pi_1(S^1)}$. There is an isomorphism

$$E_{1,1}^{2} = H_1(S^1; \mathbb{H}_1(V_g)) \cong \text{Ker}(\partial_*: H_2(M, M^{(0)}) \to H_1(M^{(0)})),$$

under which the homology class $[e^1 \otimes \alpha]$ is mapped to the homology class of the relative 2-cycle $\Phi^\sharp(I \times \tilde{a})$. However, since $e^0 \times \sigma_{\varphi, \alpha} \in S_2(M^{(0)})$, it holds that

$$[\Phi^\sharp(I \times \tilde{a})] = [\Phi^\sharp(I \times \tilde{a}) - e^0 \times \sigma_{\varphi, \alpha}] = [Z_\alpha] \in H_2(M, M^{(0)}).$$

Thus the homology class under consideration is now represented by a genuine 2-cycle in $M$.

Finally, we observe that the natural map

$$H_2(M) \cong E_{1,1}^{\infty} \xrightarrow{\cong} E_{1,1}^{2} \subset H_2(M, M^{(0)})$$

coincides with the inclusion homomorphism. This completes the proof. \qed

By Lemma 3.5, it is enough to compute $i_*([Z_\alpha])$. Since $\tilde{a}$ is a 1-cycle in $\Sigma_g = \partial V_g$, the 2-chain $\Phi^\sharp(I \times \tilde{a})$ lies in $\partial M_\varphi$. Hence

$$Z_\alpha = -e^0 \times \sigma_{\varphi, \alpha} \in Z_2(M_\varphi, \partial M_\varphi).$$

This shows that $i_*([Z_\alpha])$ is represented by the relative 2-cycle $-e^0 \times \sigma_{\varphi, \alpha}$. This completes the
proof of Proposition 3.4.

3.5. Proof of Theorem 1.1

We describe the intersection form of $M_\varphi$ and prove Theorem 1.1.

First we claim that the second homology group $H_2(M_\varphi)$ is naturally isomorphic to $U_\varphi^g := \text{Ker}(S - I_g) \subset \mathbb{Z}^g$. In fact, by Lemma 3.1 we have $H_2(M_\varphi) \cong H_1(V_g)_{\pi_1(S^1)}$, and the action of $\varphi$ on $H_1(V_g) \cong \mathbb{Z}^g$ is given by the matrix $S$. Thus the claim follows.

We next claim that under the isomorphism $H_2(M_\varphi) \cong U_\varphi^g$, the intersection form on $H_2(M_\varphi)$ is transferred to the bilinear form $\langle \cdot, \cdot \rangle_\varphi$. Since $\phi_g^V(\varphi) = \text{Sign} M_\varphi$, this will complete the proof of Theorem 1.1. The proof of this claim consists of two steps.

Step 1. We give a description of the bilinear form on $H_1(V_g)_{\pi_1(S^1)}$ that is obtained by transferring the intersection form on $H_2(M_\varphi)$. Let $\langle \cdot, \cdot \rangle_V : H_2(V_g, \Sigma_g) \times H_1(V_g) \to \mathbb{Z}$ be the intersection product of the compact oriented 3-manifold $V_g$. We have

$$\langle D_i, \beta_j \rangle_V = \delta_{ij} \text{ for any } i, j \in \{1, \ldots, g\}. \quad (3.2)$$

Let

$$H_0(S^1, \mathcal{H}_2(V_g, \Sigma_g)) \times H_1(S^1, \mathcal{H}_1(V_g)) \to \mathbb{Z} \quad (3.3)$$

be the intersection product of $H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g))$ and $H_1(S^1; \mathcal{H}_1(V_g))$ followed by the contraction of the coefficients by the form $\langle \cdot, \cdot \rangle_V$. Under the isomorphisms in Lemmas 3.1 and 3.2, this is equivalent to the intersection product $H_2(M_\varphi) \times H_2(M_\varphi, \partial M_\varphi) \to \mathbb{Z}$. By composing (3.3) and the homomorphism

$$H_1(V_g)_{\pi_1(S^1)} \times H_1(V_g)_{\pi_1(S^1)} \xrightarrow{d \otimes \text{id}} H_2(V_g, \Sigma_g)_{\pi_1(S^1)} \times H_1(V_g)_{\pi_1(S^1)} \cong H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g)) \times H_1(S^1, \mathcal{H}_1(V_g)),$$
we obtain a bilinear form on $H_1(V_g)^{\pi_1(S^1)}$. Proposition 3.4 implies that this is equivalent to the
intersection form on $H_2(M_\phi)$.

**Step 2.** We prove that the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ described in the previous
paragraph is equivalent to $\langle \ , \ \rangle_\phi$ under the identification $H_1(V_g)^{\pi_1(S^1)} \cong U_\phi^Z$. Let $x = (x_1, \ldots, x_g)$, $y = (y_1, \ldots, y_g) \in U_\phi^Z \subset \mathbb{Z}^g$. We regard $x$ as an element of $H_1(V_g)^{\pi_1(S^1)}$. Then,
we can take $\tilde{x} = \sum_{i=1}^g x_i \beta_i \in H_1(\Sigma_g)$ as a lift of $x$ which we need to compute $d(x)$. Thus we have
\[
\varphi_*(\tilde{x}) - \tilde{x} = (\alpha_1, \ldots, \alpha_g) Q^t(x_1, \ldots, x_g) = (x_1, \ldots, x_g)^t Q^t(\alpha_1, \ldots, \alpha_g),
\]
and hence $d(x) = (x_1, \ldots, x_g)^t Q^t(D_1, \ldots, D_g)$. Therefore, the pairing of $x$ and $y$ by the bilinear
form on $H_1(V_g)^{\pi_1(S^1)}$ described above is equal to
\[
\langle (x_1, \ldots, x_g)^t Q^t(D_1, \ldots, D_g), (\beta_1, \ldots, \beta_g)^t(y_1, \ldots, y_g) \rangle_V = \langle x, y \rangle_\phi.
\]

Here we used the equality (3.2). This completes the proof of Theorem 1.1.

**Remark 3.6** There is a 2-cocycle $m_\lambda$ on $\text{Sp}(2g; \mathbb{Z})$ constructed by Turaev [20] which satisfies
$[m_\lambda] = -[\tau_g] \in H^2(\text{Sp}(2g; \mathbb{Z}))$, and Walker, in page 124 of his note\(^1\), constructed a (unique)
cobounding function $j: \text{Mod}(\Sigma_g) \to \mathbb{Q}$ of the sum $\rho^* \tau_g + \rho^* m_\lambda$ of 2-cocycles. The 2-cocycle
$m_\lambda$ and the function $j$ depend on the choice of a lagrangian $\lambda \subset H_1(\Sigma_g; \mathbb{Q})$. If we choose a
suitable lagrangian $\lambda$, the restriction of $j$ to $\text{Mod}(V_g)$ is known to be a cobounding function of
$\rho^* \tau_g$, and coincides with our function $\phi^V_g$. Gilmer and Masbaum [5, Proposition 6.9] described
$j$ explicitly in a way which is similar to but different from ours.

**Remark 3.7** Since $Sy = y$ for any $y \in U_\phi$, we have $\langle x, y \rangle_\phi = \langle x, y \rangle_\phi$ for any $x, y \in U_\phi$.

Since $\langle t^t QS y$ is symmetric by (2.2), this gives a purely algebraic explanation for the symmetric
property of the form $\langle \ , \ \rangle_\phi$ on $U_\phi$.

\(^1\)K. Walker (1991). On Witten’s 3-manifold invariants, Preliminary Version [online]. Website
Remark 3.8 By Theorem 1.1, one can regard $\phi_g^V$ as a 1-cochain on $\text{urSp}(2g;\mathbb{Z})$. For $g \geq 3$, it is the unique 1-cochain which cobounds $\tau_g$ on $\text{urSp}(2g;\mathbb{Z})$ since $H^1(\text{urSp}(2g;\mathbb{Z})) = 0$; see [19, Corollary 4.4].

4. Evaluation of Meyer functions

4.1. The Meyer function on the hyperelliptic mapping class group

There is a unique 1-cochain $\phi^H_g : \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ such that for any $\varphi_1, \varphi_2 \in \mathcal{H}(\Sigma_g)$,

$$\phi^H_g(\varphi_1) + \phi^H_g(\varphi_2) - \phi^H_g(\varphi_1 \varphi_2) = \tau_g(\rho(\varphi_1), \rho(\varphi_2)). \quad (4.1)$$

The 1-cochain $\phi^H_g$ is called the Meyer function on the hyperelliptic mapping class group of genus $g$; see [4, 18].

Recall the element $s_1 = t_2t_3t_1t_2 \in \mathcal{H}(V_g) \subset \mathcal{H}(\Sigma_g)$ which was defined in Section 2.4.

Lemma 4.1 $\phi^H_g(s_1) = (2g + 3)/(2g + 1)$.

Proof Set $T_i = \rho(t_i)$ for every $i \in \{1, 2, 3\}$. Using (4.1), we have

$$\phi^H_g(s_1) = \phi^H_g(t_2) + \phi^H_g(t_3) + \phi^H_g(t_1) + \phi^H_g(t_2)$$

$$- \tau_g(T_1, T_2) - \tau_g(T_3, T_1T_2) - \tau_g(T_2, T_3T_1T_2).$$

As was shown in [4, Lemma 3.3] and [18, Proposition 1.4], we have $\phi^H_g(t_i) = (g + 1)/(2g + 1)$ for all $i \in \{1, 2, 3\}$. Also, by a direct computation we obtain $\tau_g(T_1, T_2) = 0$, $\tau_g(T_3, T_1T_2) = 0$, and $\tau_g(T_2, T_3T_1T_2) = 1$. The result follows from these equalities.

4.2. The Meyer function on the handlebody group

Recall from the introduction that we defined $\phi_g^V : \text{Mod}(V_g) \to \mathbb{Z}$ by $\varphi \mapsto \text{Sign} M_\varphi$, where $M_\varphi$ is the mapping torus of $\varphi$. 

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Lemma 4.2 The function $\phi_g^V: \text{Mod}(V_g) \to \mathbb{Z}$ cobounds the cocycle $\rho^* \tau_g$ in the handlebody group $\text{Mod}(V_g)$. If $g \geq 3$, $\phi_g^V$ is the unique cobounding function of $\rho^* \tau_g$.

Proof The uniqueness follows from the fact that $H_1(\text{Mod}(V_g))$ is torsion when $g \geq 3$.

For given two mapping classes $\varphi, \psi \in \text{Mod}(V_g)$, there is an oriented $V_g$-bundle $W(\varphi, \psi) \to P$ such that the monodromy along $\ell_1$, $\ell_2$ and $\ell_3$ are $\varphi$, $\psi$ and $(\varphi \psi)^{-1}$, respectively. The boundary of $W(\varphi, \psi)$ is written as

$$\partial W(\varphi, \psi) = E(\varphi, \psi) \cup (M_{\varphi^{-1}} \sqcup M_{\psi^{-1}} \sqcup M_{\varphi \psi}).$$

Note that $M_{\varphi^{-1}}$ is diffeomorphic to $-M_{\varphi}$ under an orientation-preserving diffeomorphism, where $-M_{\varphi}$ denotes the mapping torus $M_{\varphi}$ with orientation reversed. Since the signature of $\partial W(\varphi, \psi)$ is zero, Novikov additivity implies that

$$\text{Sign } E(\varphi, \psi) - \text{Sign } M_{\varphi} - \text{Sign } M_{\psi} + \text{Sign } M_{\varphi \psi} = 0.$$ 

This shows that $\phi_g^V$ is a cobounding function of $\rho^* \tau_g$ restricted to $\text{Mod}(V_g)$. □

Since $\dim V_{A,B} \leq 4g$ for any $A, B \in \text{Sp}(2g; \mathbb{Z})$, the signature cocycle $\tau_g$ is a bounded 2-cocycle. Therefore, it represents a class in the second bounded cohomology group $H^2_b(\text{Mod}(\Sigma_g))$.

The image of $[\tau_g]$ under the natural homomorphism $H^2_b(\text{Mod}(\Sigma_g); \mathbb{Q}) \to H^2_b(\mathcal{H}(\Sigma_g); \mathbb{Q})$ is non-trivial since the Meyer function $\phi_g^M$ is unbounded. In contrast, we have:

Proposition 4.3 Under the natural homomorphism $H^2_b(\text{Mod}(\Sigma_g); \mathbb{Q}) \to H^2_b(\text{Mod}(V_g); \mathbb{Q})$, the image of the cohomology class $[\tau_g]$ vanishes.

Proof The restriction of the signature cocycle $\tau_g$ to $\text{Mod}(V_g)$ is cobounded by the function $\phi_g^V$, and $\phi_g^V$ is a bounded function since the rank of $H_2(M_{\varphi})$ is at most $g$. □
4.3. Computation of the Meyer function on the handlebody group

Theorem 1.1 shows that the bilinear form $\langle \cdot, \cdot \rangle_\varphi$ on $U_\varphi$, whose signature coincides with $\phi^V_g(\varphi)$, can be computed from the homological monodromy $\rho(\varphi) \in \text{urSp}(2g; \mathbb{Z})$. In more detail, if

$$\rho(\varphi) = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix},$$

then $U_\varphi = \text{Ker}(S - I_g) \subset \mathbb{Q}^g$ and $\langle x, y \rangle_\varphi = t^t Q y$ for $x, y \in U_\varphi$.

The 1-cochain $\phi^V_g$, regarded as the one defined on $\text{urSp}(2g; \mathbb{Z})$, is stable with respect to $g$ in the following sense. For every non-negative integer $g \geq 0$, there is a natural embedding $\iota: \text{urSp}(2g; \mathbb{Z}) \hookrightarrow \text{urSp}(2(g+1); \mathbb{Z})$;

$$A = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \mapsto \iota(A) = \begin{pmatrix} \tilde{P} & \tilde{Q} \\ O_{g+1} & \tilde{S} \end{pmatrix},$$

where

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\phi^V_{g+1}(\iota(A)) = \phi^V_g(A)$ for any $A \in \text{urSp}(2g; \mathbb{Z})$.

Lemma 4.4 For any positive integer $m$, we have $\phi^V_g(t_1^m) = 1$.

Proof Since the action of $\rho(t_1^m)$ on $H_1(\Sigma_g)$ is given by

$$\rho(t_1): \alpha_i \mapsto \alpha_i \ (i = 1, \ldots, g), \quad \beta_1 \mapsto m\alpha_1 + \beta_1, \quad \beta_i \mapsto \beta_i \ (i = 2, \ldots, g),$$

we may assume that $g = 1$. Then $\rho(t_1^m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, and $\text{Ker}(S - I_1) = \mathbb{Z}$ on which the pairing is given by the $1 \times 1$ matrix $\begin{pmatrix} m \end{pmatrix}$. Hence $\phi^V_g(t_1^m) = 1$, as required. \hfill \Box
Lemma 4.5 \( \phi^V_g(s_1) = 1 \).

Proof The proof proceeds as in the same way as the previous lemma. In this case we may assume that \( g = 2 \). Then

\[
\rho(s_1) = \begin{pmatrix} P & Q \\ O_2 & S \end{pmatrix}
\]

with \( P = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \), \( Q = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \), \( S = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \).

The rest of computation is straightforward, so we omit it. \( \square \)

4.4. Proof of Theorem 1.2

Since both the 1-cochains \( \phi^H_g \) and \( \phi^V_g \) cobound the signature cocycle, their difference becomes a \( \mathbb{Q} \)-valued homomorphism on \( \mathcal{H}(V_g) = \mathcal{H}(\Sigma_g) \cap \text{Mod}(V_g) \).

We compare the homomorphism \( \phi^H_g - \phi^V_g \) with the generator \( \mu \in H^1(\mathcal{H}(V_g)) \) in Corollary 2.6. It is sufficient to evaluate \( \phi^H_g - \phi^V_g \) on \( s_1 \) if \( g \) is even, and on \( t_1 s_1^{\frac{g+1}{2}} \) if \( g \) is odd. By Lemmas 4.1 and 4.5 we immediately obtain

\[
(\phi^H_g - \phi^V_g)(s_1) = \frac{2}{2g + 1}.
\]  

(4.2)

This settles the case where \( g \) is even. When \( g \) is odd, we compute

\[
(\phi^H_g - \phi^V_g)(t_1 s_1^{\frac{g+1}{2}}) = (\phi^H_g - \phi^V_g)(t_1) + \frac{g + 1}{2}(\phi^H_g - \phi^V_g)(s_1)
\]

\[
= \left( \frac{g + 1}{2g + 1} - 1 \right) + \frac{g + 1}{2} \cdot \frac{2}{2g + 1}
\]

\[
= \frac{1}{2g + 1}.
\]

Here, we used the fact that \( \phi^H_g - \phi^V_g \) is a homomorphism on \( \mathcal{H}(V_g) \) in the first line; we used the fact that \( \phi^H_g(t_1) = (g + 1)/(2g + 1) \) (see the proof of Lemma 4.1), Lemma 4.4 and (4.2) in

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the second line. This completes the proof of Theorem 1.2.

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