On some sums at the $a$-points of the $k$-th derivatives of the Dirichlet $L$-functions

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Received: .201 • Accepted/Published Online: .201 • Final Version: ..201

Abstract: Let $L^{(k)}(s,\chi)$ be the $k$-th derivative of the Dirichlet $L$-function associated with a primitive character $\chi$ mod $q$ and $a$ be a complex number. The solutions of $L^{(k)}(s,\chi) = a$ are called $a$-points. In this paper, we give an asymptotic formula for the sums

$$
\sum_{\rho_{a,\chi}^{(k)}:0<\gamma_{a,\chi}^{(k)}<T} L^{(j)}(\rho_{a,\chi}^{(k)},\chi) \quad \text{and} \quad \sum_{\rho_{a,\chi}^{(k)}:1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}(\rho_{a,\chi}^{(k)},\chi) \quad \text{as} \quad T \to \infty
$$

where $j$ and $k$ are non-negative integers and $\rho_{a,\chi}^{(k)}$ denotes an $a$-point of the $k$-th derivative $L^{(k)}(s,\chi)$ and $\gamma_{a,\chi}^{(k)} = \text{Im}(\rho_{a,\chi}^{(k)})$.

This work continues the investigations of Kaptan, Karabulut & Yıldırım [7, 10] and Mazhouda & Onozuka [12].

Key words: Dirichlet $L$-function, $a$-points, value-distribution.

1. Introduction

Let $L(s,\chi)$ be the Dirichlet $L$-function associated with a primitive character $\chi$ mod $q$ and $a$ be a complex number. The zeros of $L(s,\chi) - a$, which will be denoted by $\rho_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$ are called the $a$-points of $L(s,\chi)$. First, we note that there is an $a$-point near any trivial zero $s = -2n$ if $\chi(-1) = 1$ and $s = -2n - 1$ if $\chi(-1) = -1$ for sufficiently large $n$. Apart from these $a$-points, there are only finitely many other $a$-points in the half-plane $\text{Re}(s) = \sigma \leq 0$. The $a$-points with $\beta_{a,\chi} \leq 0$ are said to be trivial. All other $a$-points lie in a strip $0 < \text{Re}(s) < A$, where $A$ is a constant depending on $a$; these numbers are called the nontrivial $a$-points. The number of these $a$-points satisfies a Riemann-von Mangoldt type formula (we refer to [14, chapter 7.2] for the proof of this formula which is stated for functions in a subclass of the Selberg class including the Dirichlet
\(L\)-functions \(L(s, \chi)\), namely
\[
N_{a, \chi}(T) = \sum_{\rho_{a, \chi} : 0 < \gamma_{a, \chi} \leq T} 1 = \frac{T}{2\pi} \log \left( \frac{qT}{2\pi c_a e} \right) + O(\log T),
\]
where \(c_a = m\) if \(a = 1\) and \(c_a = 1\) otherwise, with \(m = \min\{n \geq 2, \chi(n) \neq 0\}\). Here and in the sequel the error term depends on \(q\), however, the main term is essentially independent of \(a\). Moreover, \(N_{a, \chi}(T) \sim N_\chi(T)\) as \(T \to \infty\), where \(N_\chi(T) = N_{0, \chi}(T)\) denotes the number of nontrivial zeros \(\rho_\chi = \beta_\chi + i\gamma_\chi\) of \(L(s, \chi)\) satisfying \(0 < \gamma_\chi < T\).

In [1], Conrey and Ghosh suggested the problem of estimating the average \(\sum_{0 < \gamma^{(k)} < T} \zeta^{(j)}(\rho^{(k)})\) for non-negative integers \(j\) and \(k\), where \(\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}\) denote a zero of the \(k\)-th derivative \(\zeta^{(k)}(s)\). One of the first result on this topic was given by Fujii [3]. He gave an asymptotic formula of the sum \(\sum_{0 < \gamma < T} \zeta'(\rho)X^\beta\) for a rational number \(X > 0\). The \(k = 0\) case was treated by Kaptan, Karabulut and Yildirim in [7]. Garunkšišis and Steuding in [4] gave a generalization of Fujii’s asymptotic formula with \(X = 1\) that if \(T \to \infty\), we have
\[
\sum_{\rho_a : 0 < \gamma_a \leq T} \zeta'(\rho_a) = \left( \frac{1}{2} - a \right) \frac{T}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + (c_0 - 1 + 2a) \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right)
\]
\[
+ (1 - c_0 - c_0^2 + 3c_1 - 2a) \frac{T}{2\pi} + O \left( T e^{-C \sqrt{\log T}} \right),
\]
where \(C\) is some positive constant and \(c_n\) are the Stieltjes constants given by the Laurent series expansion of \(\zeta(s)\) at \(s = 1\),
\[
\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} c_n (s-1)^n
\]
Recently, Mazhouda and Onozuka in [12] proved that for \(j, k \in \mathbb{Z}_{\geq 0}\) and large \(T\),
\[
\sum_{1 < \gamma^{(k)} < T} \zeta^{(j)}(\rho^{(k)}) = (-1)^j (\delta_{j,0} + a\delta_{k,0} + B(j, k)) \frac{T}{2\pi} \left( \log \frac{T}{2\pi} \right)^{j+1} + O_{j,k} (T(\log T)^j),
\]
where the implicit constant in the error terms may depend on \(a\). To do so, they used the following result of Karabulut and Yildirim in [10] for fixed \(j, k \in \mathbb{Z}_{\geq 0}\) and large \(T\), one has
\[
\sum_{0 < \gamma^{(k)} < T} \zeta^{(j)}(\rho^{(k)}) = (\delta_{j,0} + B(j, k)) \frac{T}{2\pi} \left( \log \frac{T}{2\pi} \right)^{j+1} + O_{j,k} (T \log^j T),
\]
where \(\delta_{j,0} = 1\) if \(j = 0\) and \(0\) otherwise,
\[
B(j, k) = \frac{k+1}{j+1} - j \sum_{r=1}^{k} \frac{e^{-z_r}}{z_r^{j+1}} P_k(z_r) + j \sum_{r=1}^{k} \frac{1}{z_r^{j+1}},
\]
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the sum over $r$ being void in the case $k = 0$ and $z_r (r = 1, ..., k)$ being the zeros of $P_k(z) = \sum_{j=0}^{k} \frac{z^j}{j!}$.

Let $\rho_{a,\chi}^{(k)} = \beta_{a,\chi}^{(k)} + i\gamma_{a,\chi}^{(k)}$ denote an $a$-point of $L^{(k)}(s, \chi)$. Similar to the $a$-points of $L(s, \chi)$, there is an $a$-point of $L^{(k)}(s, \chi)$ near any trivial zero $s = -2n - \left(\frac{1-\chi(-1)}{2}\right)$ for sufficiently large $n$ and apart from these $a$-points, there are only finitely many other $a$-points in the half-plane $\sigma \leq C$ for any $C < 0$ (see Lemma 2.1 below).

In this paper, first we give an asymptotic formula for the sum

$$\sum_{\rho_{a,\chi}^{(k); 0 < \gamma_{a,\chi}^{(k)} < T}} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi)$$

and as a consequence, we obtain an estimate for

$$\sum_{\rho_{a,\chi}^{(k): 1 < \gamma_{a,\chi}^{(k)} < T}} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi)$$

where $a$ is a complex number. The first sum extend Karabulut and Yildirim result to the $k$-th derivative of the Dirichlet $L$-functions and is evaluated in the following theorem.

**Theorem 1.1** Let $k, j \in \mathbb{N}$ be fixed and $\chi$ be a primitive character modulo $q$. Then as $T \to \infty$, we have

$$\sum_{\rho_{a,\chi}^{(k); 0 < \gamma_{a,\chi}^{(k)} \leq T}} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi) = (-1)^j (\delta_{j,0} + B(j,k)) \frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^{j+1} + O_{j,k}\left(T (\log qT)^j\right),$$

where $B(j,k)$ is defined by (1.6).

From Theorem 1.1, we get our main result

**Theorem 1.2** Let $k, j \in \mathbb{N}$ be fixed, $a$ be a complex number and $\chi$ be a primitive character modulo $q$. Then as $T \to \infty$, we have

$$\sum_{\rho_{a,\chi}^{(k); 1 < \gamma_{a,\chi}^{(k)} \leq T}} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi) = (-1)^j (\delta_{j,0} + a\delta_{k,0} + B(j,k)) \frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^{j+1} + O_{j,k}\left(T (\log qT)^j\right)(1.10)$$

Here and in the sequel, the implicit constant in the error terms may depend on $a$.

**Remark.** By Theorem 1.2, we deduce the average value of $L^{(j)}(\rho_{a,\chi}^{(k)}, \chi)$ over the $a$-points $\rho_{a,\chi}^{(k)}$ of $L^{(k)}(s, \chi)$ with $1 < \text{Im}(\rho_{a,\chi}^{(k)}) < T$, i.e.,

$$\frac{1}{N_{k,\chi}(a,T)} \sum_{1 < \gamma_{a,\chi}^{(k)} < T} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi),$$

where $N_{k,\chi}(a,T)$ is the number of terms in the above sum. By the same argument as in [13], we have an asymptotic formula for $N_{k,\chi}(a,T)$ which is $\sim (T/2\pi) \log \frac{qT}{2\pi}$ (see [15] for the asymptotic formula of $N_{k,\chi}(0,T)$).
Hence, the average is \((-1)^j (\delta_{j,0} + a\delta_{k,0} + B(j,k)) \left( \log \frac{2\pi}{t} \right)^j \). So this tells us about the size of \( L^{(j)}(s,\chi) \) at certain points (namely the \( a \)-points of \( L^{(k)}(s,\chi) \)).

2. Preliminary lemmas and equations

In this section, we give some lemmas and formulas useful for the proof of our Theorems. We start with well-known results on the Dirichlet \( L \)-function \( L(s,\chi) \) (see Davenport book [2]) and its \( k \)-th derivative. If \( \chi \mod q \) is a primitive character, then

\[
L(s,\chi) = \Lambda(s,\chi)L(1-s,\overline{\chi}),
\]

where

\[
\Lambda(s,\chi) = \frac{2\tau(\chi)}{i^\kappa q} \left( \frac{2\pi}{q} \right)^{s-1} \Gamma(1-s) \sin \left( \frac{\pi}{2} (s + \kappa) \right),
\]

with \( \tau(\chi) = \sum_{r=1}^{q} \chi(r) e^{\frac{2\pi ir}{q}} \) and \( \kappa = \frac{1}{2}(1-\chi(-1)) \). From (2.2) and by stirling’s formula (see [9, page 13]), we get

\[
\Lambda(1-s,\chi) = \frac{\tau(\chi)}{i^\kappa \sqrt{q}} \exp \left\{ it \log \left( \frac{|t|}{2\pi e} \right) - \text{sgn}(t)(\frac{i\pi}{2} - \kappa) \right\} \left( \frac{|t|}{2\pi} \right)^{\sigma - \frac{1}{2}} \left( 1 + O \left( \frac{1}{|t|} \right) \right),
\]

in any fixed halfstrip \( \alpha \leq \sigma \leq \beta, |t| \geq 1 \). Moreover, for any fixed \( \sigma, j \geq 0 \) and \( |t| \geq 1 \), we have

\[
\frac{\Lambda'}{\Lambda}(s,\chi) = -\log \frac{|t|}{2\pi} + O \left( \frac{1}{|t|} \right), \quad \left( \frac{d}{ds} \right)^j \frac{\Lambda'}{\Lambda}(s,\chi) \ll |t|^{-j}
\]

and

\[
\Lambda^{(j)}(1-s,\chi) = \Lambda(1-s,\chi) \left( -\log \frac{|t|}{2\pi} \right)^j + O \left( q^{\sigma-\frac{1}{2}} |t|^\sigma \log^j |t| \right).
\]

Using equations (2.3), (2.4) and (2.5) with \( j \)-fold differentiation of the functional equation (2.1), we obtain

\[
L^{(j)}(1-s,\chi) = (-1)^j \Lambda(1-s,\chi) \left( 1 + O \left( \frac{1}{t} \right) \right) \sum_{m=0}^{j} \binom{j}{m} \ell^{j-m} L^{(m)}(s,\overline{\chi}),
\]
where $\sigma$ is fixed, $|t| \geq 1$ and $\ell = \log \left( \frac{q|t|}{2\pi} \right)$. Furthermore, for any fixed $\sigma, k \in \mathbb{Z}_{\geq 0}$ and $t \geq 1$, we have

$$
\frac{L^{(k+1)}}{L^{(k)}}(1-s, \chi) = -\left(1 + O \left( \frac{1}{t} \right) \right) \left( \ell + \sum_{v=0}^{k} \binom{k}{v} \frac{\ell^{v} L^{(v+1)}}{L(s, \chi)} \right)
$$

$$
= -\left(1 + O \left( \frac{1}{t} \right) \right) \left( \ell + \sum_{v=0}^{k} \binom{k}{v} \frac{L^{(v+1)}}{L(s, \chi)} \right)
$$

$$
= -\left(1 + O \left( \frac{1}{t} \right) \right) \left( \ell + \frac{G' k}{G_k} \right),
$$

(2.7)

with the differentiation in $G'$ respect to $s$. Since $L^w L(s, \chi) \ll w$ when $\sigma \geq 1 + \delta$, for sufficiently large $t$, we get

$$
\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} L^w L(s, \chi) \ll_k \frac{1}{\log qt}.
$$

(2.8)

By expanding the denominator of (2.7) as a power series, we obtain

$$
\left( 1 + \sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} L^w L(s, \chi) \right)^{-1} = \sum_{u=0}^{\infty} (-1)^u \left( \sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} L^w L(s, \chi) \right)^u
$$

$$
= \sum_{u \leq \frac{\log A}{\log A}} (-1)^u \left( \sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} L^w L(s, \chi) \right)^u + O \left( \frac{1}{A} \right)
$$

(2.9)

where $\sigma \geq 1 + \delta$ and $t \geq A$ for large $A$. By the functional equation (2.1) and the Phragmén-Lindelöf principle, we deduce that

$$
L(s, \chi) \ll \epsilon \begin{cases} |qt|^{\frac{1}{2} - \sigma + \epsilon} & \sigma < 0, \\ |qt|^{\frac{1}{2} (1 - \sigma) + \epsilon} & 0 \leq \sigma \leq 1, \\ |qt|^\epsilon & \sigma > 1, \end{cases}
$$

(2.10)

as $|t| \to \infty$ and where $\epsilon$ is an arbitrarily small positive number. Moreover, by Cauchy’s integral formula, we get

$$
L^{(k)}(s, \chi) = k! \int_{C} \frac{L(w, \chi)}{(w-s)^{k+1}} ds,
$$

where $C$ is any arbitrarily small circle centered at $s$. Using the last bound of $L(s, \chi)$, it follows that

$$
L^{(k)}(s, \chi) \ll \epsilon \begin{cases} |qt|^{\frac{1}{2} - \sigma + \epsilon} & \sigma < 0, \\ |qt|^{\frac{1}{2} (1 - \sigma) + \epsilon} & 0 \leq \sigma \leq 1, \\ |qt|^\epsilon & \sigma > 1. \end{cases}
$$

(2.11)
Now, using the same argument as in [13, Lemma 2.6], we get easily

\[
\frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi)} - a = \sum_{|\gamma_{a, \chi}| \leq 1} \frac{1}{s - \rho_{a, \chi}} + O(\log qt),
\]

(2.12)

for any constants \(\alpha, \beta\) and \(s \in \mathbb{C}\) with \(\alpha \leq \sigma \leq \beta\) and large \(t\).

**Lemma 2.1** Let \(k\) be a positive integer, \(\chi\) be a primitive character modulo \(q\) and \(a \in \mathbb{C}\). Then, there exists real numbers \(E_1 = E_1(k, a, q) \leq 0\) and \(E_2 = E_2(k, a, q) \geq 1\) such that there is no \(a\)-point of \(L^{(k)}(s, \chi)\) for \(\{s \in \mathbb{C}, \sigma \leq E_1, |t| \geq 1\}\) and \(\{s \in \mathbb{C}, \sigma \geq E_2\}\).

**Proof** The case \(a = 0\) was treated by Yildirim in [16]. Hence, we consider only the case \(a \neq 0\). From equation (2.1) and by differentiating \(k\) times, we obtain

\[
L^{(k)}(1 - s, \chi) = (-1)^k \frac{2 \tau(\chi)}{s q} \left(\frac{2 \pi}{q}\right)^{-s} \sum_{j=0}^{k} \Gamma^{(j)}(s) R_{j,k}(s)
\]

\[
= (-1)^k \frac{2 \tau(\chi)}{s q} \left(\frac{2 \pi}{q}\right)^{-s} \left\{ \Gamma^{(k)}(s) \cos \left(\frac{\pi}{2}(s - \kappa)\right) L(s, \chi) + \sum_{j=0}^{k-1} \Gamma^{(j)}(s) R_{j,k}(s) \right\},
\]

(2.13)

where

\[
R_{j,k}(s) = P_{j,k}(s) \cos \left(\frac{\pi}{2}(s - \kappa)\right) + Q_{j,k}(s) \sin \left(\frac{\pi}{2}(s - \kappa)\right),
\]

(2.14)

\[
P_{j,k}(s) = \sum_{n=0}^{k} a_{j,k,n} L^{(n)}(s, \chi)
\]

(2.15)

and

\[
Q_{j,k}(s) = \sum_{n=0}^{k} b_{j,k,n} L^{(n)}(s, \chi),
\]

(2.16)

where, \(a_{j,k,n}\) and \(b_{j,k,n}\) being constants may depend on \(q\). Using [16, Equation(13)]\textsuperscript{1}, derivatives of the Gamma function can be estimated as follows:

\[
\Gamma^{(j)}(s) = \Gamma(s) (\log s)^j \left(1 + O \left(\frac{1}{s \log s}\right)\right)
\]

(2.17)

in the region \(\{s \in \mathbb{C}, \sigma \geq 1 + \delta, |t| \geq 1\}\). Using the last estimate and the fact that in the same region \(L(s, \chi) \times 1\) and \(L^{(j)}(s, \chi) = \sum_{n \geq 2} \frac{\chi(n)(-\log n)^j}{n^s} \ll 1\), we get

\[
\left| \Gamma^{(k)}(s) \cos \left(\frac{\pi}{2}(s - \kappa)\right) L(s, \chi) \right| \ll \left| \Gamma(s) \log^k(s) e^{\pi |t|} \right|
\]

(2.18)

and

\[
\left| \sum_{j=0}^{k-1} \Gamma^{(j)}(s) R_{j,k}(s) \right| \ll \left| \Gamma(s) \log^{k-1}(s) e^{\pi |t|} \right|.
\]

(2.19)
As a consequence, one has

\[ L^{(k)}(1-s, \chi) = (-1)^{k} \frac{2\tau(\chi)}{i^s q} \left( \frac{2\pi}{q} \right)^{-s} \Gamma(s) \log^k(s) \cos \left( \frac{\pi}{2} (s - \kappa) \right) L(s, \chi) \left( 1 + O \left( \frac{1}{|\log s|} \right) \right) \]  

in the region \( \{ s \in \mathbb{C}, \sigma \geq 1 + \delta, |t| \geq 1 \} \). It follows from (2.20) that \( L^k(1-s, \chi) \to \infty \) as \( \sigma \to \infty \).

So there exists, \( E_1 = E_1(k, a, q) \leq 0 \) such that \( |L^{(k)}(s, \chi)| > |a| \) for \( \sigma \leq E_1 \) and \( |t| \geq 1 \). Next, since \( L^{(k)}(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)(-\log n)^k}{n} \) \( \to 0 \) as \( \sigma \to \infty \) there exists, \( E_2 = E_2(k, a, q) \geq 1 \) such that \( |L^{(k)}(s, \chi)| < |a| \).

\( \square \)

**Remark.** It can also be seen by Rouché’s theorem that there is \( N_k = N_k(a, q) < 0 \) such that \( L^{(k)}(s, \chi) = a \) has only one zero in the region \( \{ s \in \mathbb{C}, -1 - 2n - \kappa < \sigma < 1 - 2n - \kappa, -1 < t < 1 \} \) for \( -n < N_k \). Moreover, apart from these \( a \)-points, there are only finitely many other \( a \)-points in the half-plane \( \sigma \leq C \) for any \( C < 0 \).

From Lemma 2.1, equation (2.11) and by Jensen’s formula, we deduce easily the following lemma.

**Lemma 2.2** For any complex number \( a \) and any sufficiently large \( T \), we have

\[ N_{k, \chi}(a; 1, T + 1) - N_{k, \chi}(a; 1, T) \ll \log(qT), \]  

where

\[ N_{k, \chi}(a; 1, T) = \sum_{\rho_{k, \chi}^{(k)}: 1 < \gamma_{\rho_{k, \chi}} < T} 1. \]  

3. Proof of Theorem 1.1

To prove Theorem 1.1, we use the same argument as in [10]. For this purpose, we need to extend some lemmas for \( k \)-th derivative of Dirichlet \( L \)-function \( L_k(s, \chi) \). The case \( k = 0 \) was already proved by Kaptan, Karabulut and Yıldırım in [6], so here we assume \( k \neq 0 \).

**Lemma 3.1** Let \( (b_n)_n \) be a sequence of complex numbers such that \( b_n \ll n^\epsilon \) for any \( \epsilon > 0 \). Let \( a > 1 \) and \( m \) be an integer. Then, for \( 1 \leq T_1 \leq T \) and \( |m| = O(T) \) as \( T \to \infty \), one has

\[ \frac{1}{2\pi} \int_{T_1}^T A(1 - a - it, \chi) \left( \log \left( \frac{qt}{2\pi} \right) \right)^m \sum_{n=1}^\infty \frac{b_n}{n^{a+it}} dt = \frac{\tau(\chi)}{q} \sum_{1 \leq n \leq qT} b_n e^{-2\pi n \chi} \left( \log n \right)^m + O \left( (qT)^{a-\frac{1}{2}} (\log qT)^m \right) + O \left( q^{2a-1} (\log q)^m \right). \]

**Proof** The case \( m \) non negative is treated by Kaptan in [8, Lemma 2.14] which is based on [5, Lemma 2] (see also [10, Lemma 2.2]). For the case when \( m \) is negative, we use the same argument of Kaptan and [11, Lemma 3.5] to obtain the result.

An elementary computation yields to the following lemma.

\[ \square \]
Lemma 3.2 For \( k, i_1, i_2, \ldots, i_k, m \in \mathbb{N}, v \in \{0, 1, \ldots, k\}, \) \( \sigma > 1 \) and \( \chi \) be a Dirichlet character modulo \( q \), let define

\[
\sum_{n=1}^{\infty} \frac{c_n(i_1, i_2, \ldots, i_k; v; m; \chi)}{n^s} := \frac{L(s+1)}{L(s, \chi)}L(m)(s, \chi) \prod_{w=1}^{k} \left( \frac{L(w)}{L(s, \chi)} \right)^{i_w}.
\]

We have

\[
\chi'(n)c_n(i_1, i_2, \ldots, i_k; v; m; \chi) = c_n(i_1, i_2, \ldots, i_k; v; m; \chi')\chi,
\]

for every Dirichlet character \( \chi' \) modulo \( q \), with

\[
|c_n(i_1, i_2, \ldots, i_k; v; m; \chi)| \leq (\log n)^{K+m+1},
\]

where

\[ K := i_1 + 2i_2 + \ldots + ki_k + v. \]

Lemma 3.3 Let \( \chi \) be a Dirichlet character modulo \( q \). Let \( k, i_1, i_2, \ldots, i_k, m \in \mathbb{N}, v \in \{0, 1, \ldots, k\} \). For fixed \( k \), if \( i_1 + i_2 + \ldots + i_k \leq \frac{\log x}{\log \log x} \), then as \( T \to \infty \), we have

\[
\sum_{n \leq x} c_n(i_1, i_2, \ldots, i_k; v; m; \chi) = O_k \left(x(\log x)^{K+m}\right),
\]

if \( \chi \) is non principal and

\[
\sum_{n \leq x} c_n(i_1, i_2, \ldots, i_k; v; m; \chi) = \frac{\varphi(q)}{q} S(i_1, i_2, \ldots, i_k; v; m)x(\log x)^{K+m+1} + O_k \left(x(\log x)^{K+m}\right)
\]

if \( \chi \) is the principal character, where

\[
S(i_1, i_2, \ldots, i_k; v; m) = \frac{(-1)^{K+m+1}(v+1)!(m! \prod_{w=1}^{k} (w!)^{i_w}}{(K+m+1)!}.
\]

Proof Let \( \chi \) be a non principal character modulo \( q \). Lemma 3.2 with Perron’s formula [15, chapter 3.12], yields to

\[
\sum_{n \leq x} c_n(i_1, i_2, \ldots, i_k; v; m; \chi) = \int_{1+\frac{1}{\log x} - iU}^{1+\frac{1}{\log x} + iU} \frac{L(v+1)}{L(s, \chi)}L(m)(s, \chi) \prod_{w=1}^{k} \left( \frac{L(w)}{L(s, \chi)} \right)^{i_w} \frac{x^s}{s} ds
\]

+ \( O_k \left(\frac{x}{U}(\log x)^{K+m+2}\right) \),

where \( q \leq U \leq x \). Now, let \( C \) be the rectangle with vertices \( 1 + \frac{1}{\log x} - iU, 1 + \frac{1}{\log x} + iU, \sigma_0 + iU \) and \( \sigma_0 - iU \).

Case 1. Assume that \( L(s, \chi) \) has no exceptional zero. We take \( \sigma_0 = 1 - \frac{c}{5 \log qU} \), where \( c \) is the constant such that \( L(s, \chi) \neq 0 \) for \( \sigma > 1 - \frac{c}{5 \log qU} \) (see [2, page 93]). So, the integrand is analytic on and inside \( C \) and we have the bound \( \frac{L(w)}{L}(s, \chi) \ll (\log qU)^w \). Then, we have by Cauchy’s formula

\[
M = \frac{1}{2\pi i} \int_C \frac{L(v+1)}{L(s, \chi)}L(m)(s, \chi) \prod_{w=1}^{k} \left( \frac{L(w)}{L(s, \chi)} \right)^{i_w} \frac{x^s}{s} ds = 0.
\]
Now, using that $L^{(m)}(s, \chi) \ll (qU)^{\frac{1}{2} (1 - \sigma) + \epsilon}$, we get

$$\int_{1 - \epsilon \frac{1}{\log qU} + iU}^{1 + \epsilon \frac{1}{\log qU}} \frac{L^{(v+1)}}{L} (s, \chi) L^{(m)}(s, \chi) \prod_{u=1}^{k} \left( \frac{L^{(u)}}{L} (s, \chi) \right)^{i_{w}} \frac{x^{s}}{s} \ ds \ll (\log qU)^{K+1} \int_{1 - \epsilon \frac{1}{\log qU}}^{1 + \epsilon \frac{1}{\log qU}} (qU)^{\frac{1}{2} (1 - \sigma) + \epsilon} \frac{x^{\sigma}}{s} \ ds \sigma$$

$$\ll x (\log qU)^{K+1} \frac{x^{\sigma}}{s} \ ds \sigma \ll x U^{1 - \epsilon} (\log qU)^{K}.$$ Analogously, we have

$$\int_{1 - \epsilon \frac{1}{\log qU} - iU}^{1 + \epsilon \frac{1}{\log qU} - iU} \frac{L^{(v+1)}}{L} (s, \chi) L^{(m)}(s, \chi) \prod_{u=1}^{k} \left( \frac{L^{(u)}}{L} (s, \chi) \right)^{i_{w}} \frac{x^{s}}{s} \ ds \ll (\log qU)^{K+1} \int_{-U}^{U} (|t|)^{\frac{1}{2} (1 - \sigma_{0}) + \epsilon} \frac{x^{\sigma_{0}}}{|\sigma_{0} + it|} \ dt$$

$$\ll x (\log qU)^{K+1} \frac{x^{\sigma_{0}}}{|\sigma_{0} + it|} \ dt \ll x U^{1 - \epsilon} (\log qU)^{K}.$$ Let $U = (\log x)^{2}$. Then, from all above estimates, we obtain

$$\sum_{n \leq x} c_{n}(i_{1}, i_{2}, ..., i_{k}; v; m; \chi) = O_{k} \left( x (\log x)^{K+m} \right)$$

**Case 2.** Suppose that there is an exceptional zero $\beta$, with $\beta \geq 1 - \frac{c}{4 \log qU}$. Therefore, we take $\sigma_{0} = 1 - \frac{c}{3 \log qU}$. So, the integrand has a pole at $\beta$ of order $L+1$, where $L = i_{1} + i_{2} + ... + i_{k}$. Hence

$$M = \frac{1}{L!} \frac{d^{L}}{ds^{L}} \left\{ (s - \beta)^{L+1} \frac{L^{(v+1)}}{L} (s, \chi) L^{(m)}(s, \chi) \prod_{u=1}^{k} \left( \frac{L^{(u)}}{L} (s, \chi) \right)^{i_{w}} \frac{x^{s}}{s} \right\}_{s=\beta}$$

$$= \frac{1}{L!} \sum_{\sum_{j=1}^{L} j_{1} + j_{2} + j_{3} = L} \left\{ \frac{L}{j_{1}, j_{2}, j_{3}} \right\} \frac{d^{j_{1}}}{ds^{j_{1}}} \left\{ (s - \beta)^{L+1} \frac{L^{(v+1)}}{L} (s, \chi) L^{(m)}(s, \chi) \prod_{u=1}^{k} \left( \frac{L^{(u)}}{L} (s, \chi) \right)^{i_{w}} \frac{x^{s}}{s} \right\}_{s=\beta}$$

$$\times \frac{d^{j_{2}}}{ds^{j_{2}}} \left\{ x^{s} \right\}_{s=\beta} \frac{d^{j_{3}}}{ds^{j_{3}}} \left\{ \frac{1}{s} \right\}_{s=\beta}$$

$$= (-1)^{L} \frac{x^{\beta}}{L^{L+1}} \sum_{j_{1}=0}^{L} \frac{(-1)^{j_{1}}}{j_{1}!} \beta^{j_{1}} \frac{d^{j_{1}}}{ds^{j_{1}}} \left\{ (s - \beta)^{L+1} \frac{L^{(v+1)}}{L} (s, \chi) L^{(m)}(s, \chi) \prod_{u=1}^{k} \left( \frac{L^{(u)}}{L} (s, \chi) \right)^{i_{w}} \frac{x^{s}}{s} \right\}_{s=\beta}$$

$$\times \frac{L^{-j_{1}}}{j_{2}!} \beta^{j_{2}} (\log x)^{j_{2}}.$$
By Cauchy’s formula on a disk of radius 1 centered at \( s = \beta \), we deduce
\[
\left| \frac{d^j}{ds^j} \left( (s - \beta)^{L+1} L^{(w+1)} L^{(m)} (s, \chi) \prod_{w=1}^{k} \left( \frac{L^{(w)}}{L} (s, \chi) \right)^iw \right) \right| \leq \frac{j_1! \max_{|s-\beta|=1} \left| L^{(w+1)} L^{(m)} (s, \chi) \prod_{w=1}^{k} \left( \frac{L^{(w)}}{L} (s, \chi) \right)^iw \right|}{j_1!} \ll k^{-j_1}.
\]

The last equation yields to
\[
M \ll_k \frac{x^\beta}{\beta^{L+1}} \sum_{j_1=0}^{L} \beta^{j_1} \sum_{j_2=0}^{L-j_1} \beta^{j_2} (\log x)^{j_2} - \frac{x^\beta}{\beta} (\log x)^L - \frac{x x (\log x)^L}{\beta}.
\]

As above, we obtain
\[
\sum_{n \leq x} c_n (i_1, i_2, ..., i_k; v; m; \chi) = O_k \left( x (\log x)^{K+m} \right)
\]

**Case 3.** Suppose the existence of an exceptional zero \( \beta \), with \( \beta < 1 - \frac{c}{\log qU} \). Therefore, proceeding similarly as in case 1, we get
\[
\sum_{n \leq x} c_n (i_1, i_2, ..., i_k; v; m; \chi) = O_k \left( x (\log x)^{K+m} \right).
\]

The proof of Lemma 3.3 when \( \chi \) is principal is closely similar to that in [10, Lemma 2.4].

**Lemma 3.4** Let \( \chi \) be a Dirichlet character modulo \( q \). Let \( k, i_1, i_2, ..., i_k, m \in \mathbb{N} \) and \( v \in \{0, 1, ..., k\} \). For fixed \( k \), if \( i_1 + i_2 + ... + i_k \leq \frac{\log x}{\log \log x} \), then as \( T \to \infty \), we have
\[
\sum_{n \leq x} c_n (i_1, i_2, ..., i_k; v; m; \chi) (\log n)^{K-r} = O_{k,r,m} \left( x (\log x)^{r+m} \right)
\]

if \( \chi \) is non principal and
\[
\sum_{n \leq x} c_n (i_1, i_2, ..., i_k; v; m; \chi) (\log n)^{K-r} = \frac{\varphi(q)}{q} S(i_1, i_2, ..., i_k; v; m) x (\log x)^{r+m+1} + O_{k,r,m} \left( x (\log x)^{r+m} \right)
\]

if \( \chi \) is a principal character.

**Proof of Theorem 1.1.** The basic idea of the proof is to interpret the sum of \( L^{(j)}(\rho_{\chi}^{(k)}, \chi) \) as a sum of residues. By Cauchy’s theorem, we have
\[
\sum_{0 < \gamma_{\chi}^{(k)} < T} \int_{R} L^{(j)}(s, \chi) \frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi)} ds = \frac{1}{2\pi i} \int_{R} L^{(j)}(s, \chi) \frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi)} ds
\]
where the integration is taken over a rectangular contour in counterclockwise direction denoted by R with vertices \(-b + ic, a + ic, a + iT, -b + iT\) with some constants \(a, b, c > 0\) such that \(\frac{1}{L^{(k)}(a + it, \chi)} \ll k\), \(0 < b < \frac{1}{8}\) and \(L^{(k)}(s, \chi)\) has no zero on the lines \(t = T\) and \(t = c\). From [16, Theorem 3], we deduce that there are finitely many zeros of \(L^{(k)}(s, \chi)\) in the region \(\sigma < -b\) and \(t > c\), then we have

\[
\sum_{0 < \gamma_{\chi}^{(k)} < T} L^{(j)}(\rho^{(k)}, \chi) = \frac{1}{2\pi i} \int_{R} L^{(j)}(s, \chi) \frac{L^{(k+1)}}{L^{(k)}}(s, \chi) ds + O(1)
\]

\[
= \frac{1}{2\pi i} \left\{ \int_{a+ic}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{a+ic} \right\} L^{(j)}(s, \chi) \frac{L^{(k+1)}}{L^{(k)}}(s, \chi) ds + O(1)
\]

\[
= I_1 + I_2 + I_3 + I_4 + O(1)
\]

The first integral \(I_1\) is independent of \(T\), so \(I_1 = O(1)\). Next, we consider \(I_2\), using that \(\frac{1}{L^{(k)}(a + it, \chi)} \ll k\) and \(L^{(j)}(s, \chi) \ll 1\), we get \(I_2 = O(T)\). Now, using equation (2.12) and take the horizontal sides of the rectangular contour to be a distance \(\gg \frac{1}{\log qT}\) from any zero of \(L^{(k)}(s, \chi)\), one has

\[
I_3 = \frac{1}{2\pi i} \int_{a+iT}^{b+iT} \sum_{\gamma_{\chi}^{(k)} - t < 1} \frac{L^{(j)}(s, \chi)}{s - \rho_{\chi}^{(k)}} ds + O \left( \int_{a+iT}^{b+iT} \log(qT) L^{(j)}(s, \chi) ds \right)
\]

\[
= O \left( (qT)^{1/2} \log qT \sum_{|\gamma_{\chi}^{(k)} - T| < 1} 1 \right) + O \left( (qT)^{1/2 + b + \epsilon} \log qT \right).
\]

By Lemma 2.2, we obtain

\[
I_3 = O \left( (qT)^{1/2 + b + \epsilon} \log qT \right)^2.
\]

This leads \(I_3 \ll T\), since \(0 < b < \frac{1}{8}\). For the fourth integral \(I_4\), by using equations (2.6), (2.7) and (2.9), we obtain

\[
I_4 = -\frac{1}{2\pi i} \int_{1+b+ic}^{1+b+iT} L^{(j)}(1-s, \overline{\chi}) \frac{L^{(k+1)}}{L^{(k)}}(1-s, \overline{\chi}) ds
\]

\[
= \frac{(-1)^j}{2\pi i} \sum_{m=0}^{j} \binom{j}{m} \int_{1+b+ic}^{1+b+iT} \Lambda(1-s, \overline{\chi}) \ell^{i-m+1} L^{(m)}(s, \overline{\chi}) ds
\]

\[
+ \frac{(-1)^j}{2\pi i} \sum_{m=0}^{j} \binom{j}{m} \int_{1+b+ic}^{1+b+iT} \Lambda(1-s, \overline{\chi}) \ell^{i-m} G_k^{(s, \ell, \chi)} L^{(m)}(s, \overline{\chi}) ds + O(T)
\]

\[
= S_1 + S_2 + O(T).
\]
Lemma 3.1 gives

\[ S_1 = (-1)^j \sum_{m=0}^{j} \binom{j}{m} \sum_{1 \leq n \leq \frac{qT}{2\pi}} (-1)^m \chi(n)e^{-2\pi i n} \log n + O \left( T^{b+\frac{1}{2}} (\log qT)^{j+1} \right) \]

\[ = (-1)^j \frac{\tau(\chi)}{q} \sum_{1 \leq n \leq \frac{qT}{2\pi}} \chi(n)e^{-2\pi i n} \log n + O \left( T^{b+\frac{1}{2}} (\log qT)^{j+1} \right) \]

\[ = \begin{cases} 
O \left( T^{b+\frac{j}{2}} (\log qT)^{j+1} \right) & \text{if } j \geq 1, \\
\frac{\tau(\chi)}{q} \sum_{1 \leq n \leq \frac{qT}{2\pi}} \chi(n)e^{-2\pi i n} \log n + O \left( T^{b+\frac{j}{2}} \log qT \right) & \text{if } j = 0. 
\end{cases} \]

Recall that (see [2, page 146])

\[ e^{-2\pi i n/q} = \frac{1}{\varphi(q)} \sum_{\chi' \equiv q} \tau(\chi') \chi'(-n), \]

when \((n, q) = 1\). The last formula yields to

\[ \frac{\tau(\chi)}{q} \sum_{1 \leq n \leq \frac{qT}{2\pi}} \chi(n)e^{-2\pi i n} \log n = \frac{\tau(\chi)}{q\varphi(q)} \sum_{\chi' \equiv q} \tau(\chi') \chi'(-1) \sum_{1 \leq n \leq \frac{qT}{2\pi}} \chi(n)\chi'(n) \log n \]

\[ = \sum_{\chi' \equiv \chi} \frac{\tau(\chi)\tau(\chi') \chi'(-1)}{q\varphi(q)} \sum_{1 \leq n \leq \frac{qT}{2\pi}} \chi(n)\chi'(n) \log n \]

\[ + \frac{\tau(\chi)\tau(\chi) \chi(-1)}{q\varphi(q)} \sum_{1 \leq n \leq \frac{qT}{2\pi}} \chi_0(n) \log n. \]

Using the following estimate

\[ \sum_{1 \leq n \leq x} \chi_0(n) \log n = \frac{\varphi(q)}{q} x \log(x) + O\left( \frac{\varphi(q)}{q} x \right) + O(q^\epsilon \log(x)) \]

and Pólya-Vinogradov inequality

\[ \sum_{n \leq x} \chi(n) \ll 2\sqrt{\log q} \]

for every non principal character modulo \(q\), we obtain

\[ S_1 = \begin{cases} 
O \left( T^{b+\frac{j}{2}} (\log qT)^{j+1} \right) & \text{if } j \geq 1, \\
\frac{T}{2\pi} \log(qT/2\pi) + O \left( T^{b+\frac{j}{2}} \log qT \right) & \text{if } j = 0. 
\end{cases} \]
Now, we estimate $S_2$. We have
\[
S_2 = \frac{(-1)^j}{2\pi i} \sum_{m=0}^j \binom{j}{m} \int_{1+b+i\ell}^{1+b+iT} \Lambda(1-s, \chi) \ell^{1+\lambda+it} \frac{G_k'}{G_k} (s, \ell, \chi) L^{(m)}(s, \chi) ds + O(T)
\]
\[
= (-1)^j \sum_{m=0}^j \binom{j}{m} \sum_{u \leq \log T} \sum_{v=0}^k (-1)^u \binom{k}{v} \sum_{i_1+i_2+\ldots+i_k=u} \binom{u}{i_1, i_2, \ldots, i_k} \prod_{w=1}^k \binom{k}{w} i_w
\]
\[
\times \frac{1}{2\pi} \int_c \Lambda(-b-it, \chi) \ell^{1-K-m} L^{(m)}(1+b+it, \chi) \frac{L^{(v+1)}}{L}(1 + b + it, \chi) \prod_{w=1}^k \left( \frac{L(w)}{L}(1 + b + it, \chi) \right)^i_w dt
\]
\[
+ O_{j,k} \left( T^{\frac{3}{2} + b + \epsilon} \right).
\]

From Lemma 3.1, we get
\[
S_2 = (-1)^j \sum_{m=0}^j \binom{j}{m} \sum_{u \leq \log T} \sum_{v=0}^k (-1)^u \binom{k}{v} \sum_{i_1+i_2+\ldots+i_k=u} \binom{u}{i_1, i_2, \ldots, i_k} \prod_{w=1}^k \binom{k}{w} i_w
\]
\[
\times \frac{\tau(\chi)}{q} \sum_{1 \leq n \leq \frac{T}{2\pi}} C_n(i_1, i_2, \ldots, i_k; v; m; \chi) e^{-\frac{2\pi in}{\chi}} \sum_{\chi' \equiv q} \frac{1}{\varphi(q)} \tau(\chi') \chi'(-n)
\]

Since
\[
e^{-\frac{2\pi in}{q}} = \frac{1}{\varphi(q)} \sum_{\chi' \equiv q} \tau(\chi') \chi'(-n)
\]
when $(n, q) = 1$, we obtain
\[
S_2 = (-1)^j \sum_{m=0}^j \binom{j}{m} \sum_{u \leq \log T} \sum_{v=0}^k (-1)^u \binom{k}{v} \sum_{i_1+i_2+\ldots+i_k=u} \binom{u}{i_1, i_2, \ldots, i_k} \prod_{w=1}^k \binom{k}{w} i_w
\]
\[
\times \left\{ \sum_{\chi' \neq \chi} \frac{\tau(\chi') \chi'(-1)}{q \varphi(q)} \sum_{1 \leq n \leq \frac{T}{2\pi}} \chi'(-n) c_n(i_1, i_2, \ldots, i_k; v; m; \chi) \right\}
\]
\[
+ O_{j,k} \left( T^{\frac{3}{2} + b + \epsilon} \right).
\]

By Lemma 3.4, we deduce
\[
S_2 = (-1)^j \frac{T}{2\pi} \left( \log \frac{qT}{2\pi} \right)^{j+1} \sum_{m=0}^j \binom{j}{m} \sum_{u \leq \log T} \sum_{v=0}^k (-1)^u \binom{k}{v}
\]
\[
\times \sum_{i_1+i_2+\ldots+i_k=u} \binom{u}{i_1, i_2, \ldots, i_k} \prod_{w=1}^k \binom{k}{w} i_w \frac{(-1)^{K+m+1}(v+1)!m!}{(K+m+1)!} + O_{j,k} \left( T \log qT \right)^j
\]

This last sum $S_2$ was evaluated by Karabulut and Yildirim in [10]
\[
S_2 = (-1)^j \frac{T}{2\pi} \left( \log \frac{qT}{2\pi} \right)^{j+1} B(j, k) + O_{j,k} \left( T \log qT \right)^j
\]
Combining $S_1$ and $S_2$, we obtain

$$I_4 = (-1)^j (\delta_{j,0} + B(j,k)) \frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^{j+1} + O_{j,k} \left(T (\log qT)^j\right).$$

Finally, theorem 1.1 follows from estimates of $I_1, I_2, I_3$ and $I_4$.

4. Proof of Theorem 1.2

Let $a$ be a complex number. We write $s = \sigma + it$, $\rho^{(k)}_{a,\chi} = \beta^{(k)}_{a,\chi} + i\gamma^{(k)}_{a,\chi}$ with real numbers $\sigma, t, \beta^{(k)}_{a,\chi}$ and $\gamma^{(k)}_{a,\chi}$. The case $a = 0$ was already proved in Theorem 1.1, so here we assume $a \neq 0$. By the residue theorem, for a sufficiently large constant $B$ and constant $b \in (1, 9/8)$, we have

$$\sum_{1 < \gamma^{(k)}_{a,\chi} < T} L^{(j)} \left(\rho^{(k)}_{a,\chi}, \chi\right) = \frac{1}{2\pi i} \int_{\mathbb{R}} L^{(j)}(s, \chi) \frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi) - a} \, ds + O(1),$$

where the integration is taken over a rectangular contour in counterclockwise direction denoted by $\mathbb{R}$ with vertices $1 - b + i, B + i, B + iT, 1 - b + iT$. Since there are finitely many $a$-points in $\{s \in \mathbb{C}; \text{Re}(s) \leq 1 - b, \text{Im}(s) \geq 1\}$, we have

$$\sum_{1 < \gamma^{(k)}_{a,\chi} < T} L^{(j)} \left(\rho^{(k)}_{a,\chi}, \chi\right) = \frac{1}{2\pi i} \int_{\mathbb{R}} L^{(j)}(s, \chi) \frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi) - a} \, ds + O(1).$$

Hence,

$$\sum_{1 < \gamma^{(k)}_{a,\chi} < T} L^{(j)} \left(\rho^{(k)}_{a,\chi}, \chi\right) = \frac{1}{2\pi i} \left\{ \int_{1-b+i}^{B+i} + \int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i} \right\} L^{(j)}(s, \chi) \frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi) - a} \, ds + O(1)$$

$$:= \frac{1}{2\pi i} (I_1 + I_2 + I_3 + I_4) + O(1).$$

The integral $I_1$ is independent of $T$, so we have $I_1 = O(1)$. Next, we consider $I_2$. Since $L^{(k)}(s, \chi) \to 0$ as $\sigma \to \infty$ if $k \geq 1$, we choose in this case $B$ such that $|L^{(k)}(B + it, \chi)| < \frac{|t|}{\pi}$, then we have $\frac{1}{L^{(k)}(B + it, \chi) - a} \ll k / 1$. Using this and $L^{(j)}(s, \chi) \ll 1$, we get

$$I_2 = O(T).$$

For the case $k = 0$, recall that, for $\sigma \to \infty$, we have $L(s, \chi) = 1 + o(1)$ and $L'(s, \chi) \ll 2^{-\sigma}$ uniformly in $t$. Hence, there are no $a$-points for sufficiently large $\sigma$ provided that $a \neq 1$. For the case $a = 1$, we define $m = \min\{n \geq 2, \chi(n) \neq 0\}$. We observe, for $\sigma \to \infty$, $L(s, \chi) - 1 = \frac{\chi(m)}{m^{\sigma-1}} (1 + o(1))$. Hence, we choose $B$ a fixed constant sufficiently large such that there are no $a$-points of $L(s, \chi)$ in the half-plane $\sigma > B - 1$. Therefore, we deduce that

$$I_2 = O(T).$$
From equation (2.12), we get

\[ I_3 = \sum_{|\gamma^{(k)}_{a,\chi} - T| < 1} \int_{B + iT}^{1 - b + iT} \frac{L^{(j)}(s, \chi)}{s - \rho^{(k)}_{a,\chi}} ds + O \left( \int_{B + iT}^{1 - b + iT} (\log qt)L^{(j)}(s, \chi) ds \right). \]

Now, we change the path of integration. If \( \gamma^{(k)}_{a,\chi} < T \), we change the path to the upper semicircle with center \( \rho^{(k)}_{a,\chi} \) and radius 1. If \( \gamma^{(k)}_{a,\chi} > T \), we change the path to the lower semicircle with center \( \rho^{(k)}_{a,\chi} \) and radius 1. Then, we have

\[ \frac{1}{s - \rho^{(k)}_{a,\chi}} \ll 1 \]

on the new path. This estimate and the bound (21) yields

\[ I_3 = O \left( (qT)^{b - \frac{1}{2} + \epsilon} \sum_{|\gamma^{(k)}_{a,\chi} - T| < 1} 1 \right) + O \left( (qT)^{b - \frac{1}{2} + \epsilon} \log \log qT \right). \]

By Lemma 2.2, we obtain

\[ I_3 = O \left( (qT)^{b - \frac{1}{2} + \epsilon} \log \log qT \right). \]

This leads \( I_3 \ll T \), since \( 1 < b < 9/8 \).

Finally, we estimate \( I_4 \). By equation (2.20) and Stirling’s formula, for fixed \( 1 < b < 9/8 \) and large \( |t| > 2 \), we have

\[ \left| L^{(k)}(1 - b + it, \chi) \right| \asymp |qt|^{b-1/2} |\log |t||^k. \] (4.3)

Therefore, there exists a constant \( A \) such that

\[ \left| \frac{a}{L^{(k)}(1 - b + it, \chi)} \right| < 1 \]

holds for any \( |t| \geq A \). We divide the path of the integral into two parts

\[ I_4 = \left( \int_{1 - b + iT}^{1 - b + iT} + \int_{1 - b + iT}^{1 - b + iT} \right) L^{(j)}(s, \chi) \frac{L^{(k+1)}(s, \chi)}{L^{(k)}(s, \chi)} ds. \]

The second term is \( O(1) \) since it is independent of \( T \). Since the integrand of the first term has a geometric series, we have

\[ I_4 = - \sum_{n=0}^{\infty} a^n \int_{1 - b + iT}^{1 - b + iT} L^{(j)}(s, \chi) L^{(k+1)}(s, \chi) (L^{(k)}(s, \chi))^{n+1} ds + O(1). \]

By (4.3), the integrand can be estimated as

\[ \frac{L^{(j)}(s, \chi)L^{(k+1)}(s, \chi)}{(L^{(k)}(s, \chi))^{n+1}} \asymp |qt|^{(b-1/2)(1-n)}(\log t)^{-kn+j+1}. \] (4.4)
Hence, each integral can be calculated as

\[ I_{4} = - \int_{1-b+iA}^{1-b+iT} \frac{L(j)(s, \chi)L^{(k+1)}(s, \chi)}{(L^{(b)}(s, \chi))^{n+1}} ds \ll (qT)^{(b-1/2)(1-n)+1+\varepsilon} \]

for any small \( \varepsilon > 0 \). It follows from the last estimate that the sum for \( n \geq 2 \) is bounded as

\[ \sum_{n=2}^{\infty} a^{n} \int_{1-b+iA}^{1-b+iT} \frac{L(j)(s, \chi)L^{(k+1)}(s, \chi)}{(L^{(b)}(s, \chi))^{n+1}} ds \ll T^{-(b-1/2)+1+\varepsilon} \ll T^{1/2}. \]

Therefore, we get

\[ I_{4} = - \int_{1-b+iA}^{1-b+iT} \frac{L(j)(s, \chi)L^{(k+1)}(s, \chi)}{L^{(b)}(s, \chi)} ds - a \int_{1-b+iA}^{1-b+iT} \frac{L(j)(s, \chi)L^{(k+1)}(s, \chi)}{(L^{(b)}(s, \chi))^{2}} ds + O(T^{1/2}) \]
\[ := -K_{1} - aK_{2} + O\left(T^{1/2}\right). \]

We already studied \( K_{1} \) in Theorem 1.1 and we get the estimate

\[ K_{1} = -2\pi i \left\{ \delta_{j,0} \frac{T}{2\pi} \log \frac{qT}{2\pi} + (-1)^{j} B(j, k) \frac{T}{2\pi} \left( \log \frac{T}{2\pi} \right)^{j+1} + O(T(\log qT)^{j}) \right\}. \]

It remains to evaluate \( K_{2} \). By equation (4.4), for \( k \geq 1 \), one has

\[ K_{2} \ll \int_{1-b+iA}^{1-b+iT} |\log t|^{j} |ds| \ll T(\log T)^{j}. \]

In the case \( k = 0 \), we use equations (2.1) and (2.6) to obtain

\[ \frac{L(j)(s, \chi)L'(s, \chi)}{L^{2}(s, \chi)} = (-1)^{j+1} \ell^{j+1} \left( 1 + O\left( \frac{1}{|t|} \right) \right) \]

(4.5)

for fixed \( \sigma \) and \( |t| \gg 1 \), where \( \ell := \log(q|t|)/2\pi \). Then, we have

\[ K_{2} = \int_{1-b+iA}^{1-b+iT} \left( (-\ell)^{j+1} + O(\log q|t|^{j}) \right) ds \]
\[ = (-1)^{j+1} iT \left( \log \frac{qT}{2\pi} \right)^{j+1} + O(T(\log qT)^{j}). \]

Combining estimates of \( K_{1} \) and \( K_{2} \), we get

\[ I_{4} = (-1)^{j} 2\pi i \left( \delta_{j,0} + a\delta_{k,0} + B(j, k) \right) \frac{T}{2\pi} \left( \log \frac{qT}{2\pi} \right)^{j+1} + O(T(\log qT)^{j}). \]

Finally, Theorem 1.2 follows from estimates of \( I_{1}, I_{2}, I_{3} \) and \( I_{4} \).

5. Concluding remarks

The \( \alpha \)-points of an \( L \)-function \( L(s) \) are the roots of the equation \( L(s) = \alpha \). We refer to Steuding book [14, chapter 7] for some results about \( \alpha \)-points of \( L \)-functions from the Selberg class. Therefore, it is an interesting question to extend Theorem 1.1 and mainly Theorem 1.2 to the other class of Dirichlet \( L \)-functions (the Selberg class with some further condition) and its higher derivative. This problem will be considered in a sequel to this paper since it is done for the Riemann Zeta function and its \( k \)-th derivative in [6] and [12].
Acknowledgment

The authors would like to express their sincere gratitude to the referee for her/his many valuable suggestions which increased the clarity of the presentation.

References