On the $n$-strong Drazin invertibility in rings

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Abstract: Let $R$ be a ring and $n$ be a positive integer. In this paper, further results on the $n$-strong Drazin inverse are obtained in a ring. We prove that $a \in R$ is $n$-strongly Drazin invertible if and only if $a-a^{n+1}$ is nilpotent. In terms of this characterization, the extensions of Cline’s formula and Jacobson’s lemma for this inverse are proved. Moreover, the $n$-strong Drazin invertibility for the sums of two elements is considered. We prove that $a, b \in R$ are $n$-strongly Drazin invertible if and only if $a+b$ is $n$-strongly Drazin invertible, under the condition $ab = 0$. As applications for the additive results, we obtain some equivalent conditions of the $n$-strong Drazin invertibility of matrices over a ring.

Key words: Strong Drazin inverse, Hirano inverse, $n$-strong Drazin inverse, Drazin inverse ring

1. Introduction

Let $R^D$ denote the set of all Drazin invertible elements in a ring $R$. It is well known that if $a,b \in R$, then

$$ab \in R^D \iff ba \in R^D.$$  

In this case, $(ba)^D = b((ab)^D)^2a$ [4]. This formula is called Cline’s formula for the Drazin inverse. Many researchers considered Cline’s formula for various types of generalized inverses, such as $(b,c)$-inverse [10], Mary inverse [27], Hirano inverse [2], pseudo-Drazin inverse [20], generalized Drazin inverse [13, 14, 16, 23, 24]. In [23], Zeng et al. extended Cline’s formula for the (pseudo, generalized) Drazin inverse to more general case. Namely, if $a,b,c,d \in R$ satisfy $acd = dbd$ and $dba = aca$, then

$$ac \in R^D \iff bd \in R^D.$$  

In this case, $(bd)^D = b((ac)^D)^2d$ and $(ac)^D = d((bd)^D)^3bac$. Corresponding to Cline’s formula, many researchers paid attention to Jacobson’s lemma, that is

$$1-ab \in R^{-1} \iff 1-ba \in R^{-1}.$$  

In this case, $(1-ba)^{-1} = 1+b(1-ab)^{-1}a$. They investigated Jacobson’s lemma for different generalized inverses in different settings [1, 2, 5, 6, 17, 18, 25].

The topic for generalized inverses of the sums was studied by many authors. In 1958, Drazin [9] proved that $a+b \in R^D$ with $(a+b)^D = a^D + b^D$ under the condition $a,b \in R^D$ and $ab = ba = 0$. For $a,b \in \mathbb{C}^{n \times n}$,

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Hartwig et al. [12] obtained a formula for \((a + b)^D\) under the one-sided condition \(ab = 0\), which was extended to the additive category by Chen et al. [3]. In addition, the problem of generalized inverses of \(a + b\) was also studied under the condition \(ab = ba\). For example, Wei and Deng [21] gave the relations of Drazin inverses of \(a + b\) and \(1 + a^Db\), where \(a, b \in \mathbb{C}^{n \times n}\). Later, Zhuang et al. [26] extended the result of [21] to the ring case. The generalized Drazin invertibility and strong Drazin invertibility of the sums under the commutative condition were also investigated [7, 8, 19].

All results mentioned above were the motivation for further consideration of the \(n\)-strong Drazin inverse in a ring. This article consists of five sections. In Section 2, we recall the definitions of some generalized inverses and give related notations. In Section 3, characterizations of \(n\)-strongly Drazin invertible elements are given in terms of the nilpotency. Then, we investigate Cline’s formula and Jacobson’s lemma for the \(n\)-strongly Drazin inverse in a ring. In Section 4, we obtain some equivalent conditions for the \(n\)-strong Drazin invertibility of the sum \(a + b\) under the hypothesis \(ab = 0\) (or \(a^2b = aba, ab^2 = bab\)). In Section 5, as applications of the previous additive results, we mainly consider the \(n\)-strong Drazin invertibility of matrices over a ring. We remark that some results presented in this paper are different from those of Drazin inverses.

2. Preliminaries

Throughout this paper, \(R\) denotes a ring with unity 1. \(R^{nil}\) and \(\mathbb{N}\) stand for the sets of all nilpotent elements in \(R\) and positive integers, respectively. Denote by \(\binom{n}{k}\) the binomial coefficient \(\frac{n!}{k!(n-k)!}\) \((0 \leq k \leq n)\).

For the readers’ convenience, we first recall the definitions of some generalized inverses. The Drazin inverse [9] of \(a \in R\) is the element \(x \in R\) which satisfies

\[
xa = x, \quad ax = xa, \quad \text{and} \quad a - a^2x \in R^{nil}.
\]

The element \(x\) above is unique if it exists and is denoted by \(a^D\). The power of nilpotency of \(a - a^2a^D\) is called the index of \(a\), and will be denoted by \(\text{ind}(a)\). Drazin [9] proved that \(a \in R\) is Drazin invertible if and only if \(a\) is both right \(\pi\)-regular (i.e. \(a^m \in a^{m+1}R\), for some \(m \in \mathbb{N}\)) and left \(\pi\)-regular (\(a^m \in Ra^{m+1}\), for some \(m \in \mathbb{N}\)), namely \(a\) is strongly \(\pi\)-regular.

In 2017, Wang [19] gave the notion of the strong Drazin inverse in a ring. An element \(a \in R\) is said to be strongly Drazin invertible [19] if there exists \(x \in R\) such that

\[
xa = x, \quad ax = xa, \quad \text{and} \quad a - ax \in R^{nil}.
\]

In this case, \(x\) is unique if it exists and is called the strong Drazin inverse of \(a\). We will denote the strong Drazin inverse of \(a\) by \(a^sD\). The strongly Drazin invertible elements are exactly the ones which are strongly nil-clean (see [19, Lemma 2.2]). Let \(a \in R\), then \(a^D\) exists if and only if there exists \(x \in R\) such that

\[
x \in aR \cap Ra, \quad ax = xa, \quad \text{and} \quad a - ax \in R^{nil}.
\]

Suppose that \(a^D\) exists. Then, let \(x = aa^D\). Obviously, \(x\) satisfies \(x \in aR \cap Ra, \ ax = xa, \ \text{and} \ a - ax \in R^{nil}\).

On the contrary, we have \((a - ax)^m = 0\) for some \(m \in \mathbb{N}\). Hence, \(a^m(1 - x)^m = a^m(1 + \sum_{i=1}^{m}(-1)^i \binom{m}{i} x^i) = 0\), which implies that \(a^m = a^m xu = uxa^m\), for some \(u \in R\). Observe that \(x = as = ta\), where \(s, t \in R\). Hence, we deduce that \(a^m = a^{m+1}su = uta^{m+1} \in a^{m+1}R \cap Ra^{m+1}\). Hence, \(a^D\) exists.
The definition of the Hirano inverse \cite{2} was introduced by Chen and Sheibani in 2017. The Hirano inverse of \( a \in R \) is the unique element \( x \) (written \( x = a^H \)) satisfying
\[
  xax = x, \quad ax = xa, \quad \text{and} \quad a^2 - ax \in R^{nil}.
\]
It is interesting that the Hirano inverse is related to tripotent elements (see \cite[Theorem 3.3]{2}). In addition, they obtained the relations of the above three kinds of generalized inverses, that is \( R^{sD} \subseteq R^H \subsetneq R^D \), where \( R^{sD} \) and \( R^D \) mean the sets of all strongly Drazin invertible and Hirano invertible elements in \( R \), respectively.

Recently, motivated by the concepts of the strong Drazin inverse and Hirano inverse, Mosić \cite{15} introduced the notion of the \( n \)-strong Drazin inverse in a ring. Let \( n \in \mathbb{N} \). An element \( x \in R \) is called the \( n \)-strong Drazin inverse of \( a \in R \) if it satisfies
\[
  xax = x, \quad ax = xa, \quad \text{and} \quad a^n - ax \in R^{nil}.
\]
The previous \( x \) is unique if such element exists, and we denote it by \( a^{nsD} \). Clearly, the \( n \)-strong Drazin inverse covers the strong Drazin inverse and Hirano inverse, that is, \( a^{1sD} = a^{sD} \) and \( a^{2sD} = a^H \). The power of nilpotency of \( a^n - aa^{nsD} \) is called the \( n \)-strong Drazin index of \( a \), denoted by \( n\text{-ind}(a) \). The symbol \( R^{nsD} \) denote the set of all \( n \)-strongly Drazin invertible elements in \( R \). We note that \( R^{nil} \subseteq R^{nsD} \). Indeed, \( a \in R^{nil} \) if and only if \( a \in R^{nsD} \) with \( a^{nsD} = 0 \). In addition, \( R^{-1} \nsubseteq R^{nsD} \). For example, let \( R = \mathbb{C} \). Then, \( 3 \in R^{-1} \), but \( 3 \notin R^{nsD} \).

Next, we introduce two known lemmas, which are related to the nilpotency.

**Lemma 2.1** Let \( a, b \in R \) with \( ab = ba \). Then,
1. If \( a \in R^{nil} \) (or \( b \in R^{nil} \)), then \( ab \in R^{nil} \).
2. If \( a, b \in R^{nil} \), then \( a + b \in R^{nil} \).

**Lemma 2.2** \cite[Lemma 3.5]{22} Let \( a \in R \). If \( a^2 - a \in R^{nil} \), then there exists a monic polynomial \( \theta(t) \in \mathbb{Z}[t] \) such that \( \theta(a) = \theta(a)^2 \) and \( a - \theta(a) \) is nilpotent.

### 3. Cline’s formula and Jacobson’s lemma

In this section, we give an existence criterion for the \( n \)-strong Drazin inverse in a ring. Then, by this characterization we prove Cline’s formula and Jacobson’s lemma for the \( n \)-strong Drazin inverse. The results presented extend the corresponding ones of the strong Drazin inverse \cite{19} and Hirano inverse \cite{2}.

Firstly, we give the relationship between the \( n \)-strong Drazin inverse and Drazin inverse. The proof of the following proposition is similar to that of \cite[Lemma 2.1]{15}.

**Proposition 3.1** Let \( n \in \mathbb{N} \). If \( a \in R^{nsD} \) with \( n\text{-ind}(a) = m \), then \( a \in R^D \) and \( a^D = a^{nsD} \). Moreover, \( \text{ind}(a) \leq nm \).

**Proof** Assume that \( a \in R^{nsD} \) with \( n\text{-ind}(a) = m \). Let \( x = a^{nsD} \). Then we have \( xax = x, ax = xa, \) and \( (a^n - ax)^m = 0 \), which yield

\[
  (a - a^2 x)^{nm} = (a^n - a^{n+1} x)^m = (a^n - ax)^m (1 - ax)^m = 0.
\]
Remark 3.3

Theorem 3.2

Let \( n \in \mathbb{N} \). Then \( a \in R^{nsD} \) if and only if \( a - a^{n+1} \in R^{nil} \).

Proof

Suppose that \( a \in R^{nsD} \) and \( x = a^{nsD} \), i.e. \( xax = x, ax = xa \), and \( a^n - ax \in R^{nil} \). Then we deduce that

\[
a^n - a^{2n} = (a^n - ax)(1 - ax - a^n) \in R^{nil},
\]

which yields that

\[
(a - a^{n+1})^n = (a - a^{n+1})a^{n-1}(1 - a^n)^{n-1} = (a^n - a^{2n})(1 - a^n)^{n-1} \in R^{nil}.
\]

Hence, \( a - a^{n+1} \in R^{nil} \).

On the contrary, since \( a - a^{n+1} \in R^{nil} \), we conclude that \( (a^n)^2 - a^n = a^{n-1}(a^{n+1} - a) \in R^{nil} \). By Lemma 2.2, there exists a monic polynomial \( \theta(t) \in \mathbb{Z}[t] \) such that \( \theta(a^n) = \theta(a^{n-1}) \) and \( a^n - \theta(a^n) \in R^{nil} \). Take \( e = \theta(a^n) \). Then we have \( e = e^2, ce = ae \) and \( a^n - e \in R^{nil} \). Hence, we obtain \( 1 + a^n - e \in R^{-1} \).

Let \( x = (1 + a^n - e)^{-1}a^{n-1}e \). Next, we show that \( a^{nsD} = x \) by the definition of the \( n \)-strong Drazin inverse. Obviously, \( ax = xa \). Note that \( a^ne = (1 + a^n - e)e = e(1 + a^n - e) \). Then, we obtain

\[
xax = (1 + a^n - e)^{-1}a^{n}e(1 + a^n - e)^{-1}a^{n-1}e = (1 + a^n - e)^{-1}(1 + a^n - e)\epsilon(1 + a^n - e)^{-1}a^{n-1}e = (1 + a^n - e)^{-1}a^{n-1}e = x
\]

and

\[
a^n - ax = a^n - (1 + a^n - e)^{-1}a^ne = a^n - e \in R^{nil}.
\]

Therefore, \( a \in R^{nsD} \) with \( a^{nsD} = x \).

Remark 3.3

(1) Let \( A \in \mathbb{C}^{m \times m} \) (rank \( A = r > 0 \)) have the Jordan form

\[
A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1},
\]

where \( D \) is invertible and \( N \) is nilpotent. Then, by Theorem 3.2 we have

\[
A \in (\mathbb{C}^{m \times m})^{nsD} \iff I - D^n \in (\mathbb{C}^{r \times r})^{nil} \iff \sigma(A) \subseteq \{0, 1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{n-1}\},
\]

where \( \sigma(A) \) denotes the spectrum of \( A \) and \( \varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \).

(2) We have the following special case,

\[
a \in R^{nsD} \text{ and } n\text{-ind}(a) = 1 \iff a^n = a^{2n}.
\]
The necessity is obvious. In fact, from \((a^n - aa^{nsD})^1 = 0\) it follows that \(a^n = aa^{nsD}\) and consequently \(a^{2n} = aa^{nsD}aa^{nsD} = aa^{nsD} = a^n\). On the contrary, suppose that \(a^n = a^{2n} \). Let \(x = a^{2n-1} \). Then, we have \(xax = a^{2n}a^{2n-1} = a^n a^{2n-1} = a^{2n} a^{n-1} = a^{2n-1} = x\). In addition, it is clear that \(ax = xa\) and \(a^n - ax = 0\). Hence, we get \(a \in R^{nsD}\) and \(a^{nsD} = x\). Moreover, \(n\text{-ind}(a) = 1\).

Applying Theorem 3.2, we get some properties of the \(n\)-strong Drazin inverse in a ring.

**Corollary 3.4** Let \(n, k \in \mathbb{N}\). If \(a \in R^{nsD}\), then \(a^k \in R^{nsD}\) and \((a^k)^{nsD} = (a^{nsD})^k\).

**Proof** Since \(a \in R^{nsD}\), by Theorem 3.2 we have \(a - a^{n+1} \in R^{nil}\), which yields

\[
a^k - (a^k)^{n+1} = a^k - (a^{n+1})^k = (a - a^{n+1}) \sum_{i=0}^{k-1} a^{n+k-1} \in R^{nil}.
\]

Hence, we obtain \(a^k \in R^{nsD}\). In view of Proposition 3.1 and [9, Theorem 2], one can see that \((a^k)^{nsD} = (a^k)^D = (a^D)^k = (a^{nsD})^k\).

**Corollary 3.5** Let \(n \in \mathbb{N}\). If \(a \in R^{nsD}\), then \(a^{nsD} \in R^{nsD}\) and \((a^{nsD})^{nsD} = a^2 a^{nsD}\).

**Proof** Let \(x = a^{nsD}\). Then we get \(xax = x\), \(ax = xa\), and \(a^n - ax \in R^{nil}\). Hence,

\[
x - x^{n+1} = x^{n+1}(a^n - ax) \in R^{nil}.
\]

Hence, by Theorem 3.2 we obtain \(x \in R^{nsD}\). From [9, Theorem 3], it follows that

\[
x^{nsD} = x^D = (a^D)^D = a^2 a^D = a^2 a^{nsD}.
\]

**Corollary 3.6** Let \(n \in \mathbb{N}\). Then,

(1) If \(a \in R^D\), then \(a \in R^{nsD}\) and \(a^{nsD} = a^D = a^D\).

(2) If \(a \in R^H\), then \(a \in R^{2nsD}\) and \(a^{2nsD} = a^H = a^D\).

(3) If \(a \in R^{nsD}\), then \(a \in R^{2nsD}\) and \(a^{2nsD} = a^{nsD} = a^D\).

**Proof** (1) Since \(a \in R^D\), by Theorem 3.2 we have \(a - a^2 \in R^{nil}\), which gives

\[
a - a^{n+1} = a(1 - a^n) = a(1 - a) \sum_{i=0}^{n-1} a^i = (a - a^2) \sum_{i=0}^{n-1} a^i \in R^{nil}.
\]

Hence, \(a - a^{n+1} \in R^{nil}\), i.e. \(a \in R^{nsD}\). In addition, \(a^{nsD} = a^D = a^D\).

(2) can be proved in the same way as the item (1).

(3) follows directly by the equality \(a - a^{2n+1} = (a - a^{n+1})(1 + a^n)\).

In terms of Theorem 3.2, we are now in the position to prove the extension of Cline’s formula for the \(n\)-strong Drazin inverse when \(acd = dbd\) and \(dba = aca\).
Theorem 3.7 Let \( a, b, c, d \in R \) and \( n \in \mathbb{N} \). If \( acd = dbd \) and \( dba = aca \), then

\[
ac \in R^{nsD} \iff bd \in R^{nsD}.
\]

In this case, \((bd)^{nsD} = b((ac)^{nsD})^{2}d \) and \((ac)^{nsD} = d((bd)^{nsD})^{3}bac\).

**Proof** It will suffice to prove the sufficiency, since the necessity can be proved similarly. From \( dba = aca \), it follows that \((ac)^{i} = (db)^{i-1}ac \) for any \( i \in \mathbb{N} \). Now, we show that

\[
(ac - (ac)^{n+1})^{m+1} = d(bd - (bd)^{n+1})^{m-1}(b - (bd)^{n}b)(ac - (ac)^{n+1})
\]

by induction on positive integer \( m \).

For \( m = 1 \), we have

\[
(ac - (ac)^{n+1})^{2} = (ac - (ac)^{n+1})(ac - (ac)^{n+1})
= ((ac)^{2} - (ac)^{n+2})(1 - (ac)^{n})
= (dbac - (db)^{n+1}ac)(1 - (ac)^{n})
= (db - (db)^{n+1})(ac - (ac)^{n+1})
= d(b - (bd)^{n}b)(ac - (ac)^{n+1}).
\]

Assume that the conclusion holds for positive integer \( m = l \). Now, we check it for \( m = l + 1 \) as follows:

\[
(ac - (ac)^{n+1})^{l+2} = (ac - (ac)^{n+1})^{l+1}(ac - (ac)^{n+1})
= d(bd - (bd)^{n+1})^{l+1}(b - (bd)^{n}b)(ac - (ac)^{n+1})^{2}
= d(bd - (bd)^{n+1})^{l}(b - b(db)^{n})d(b - (bd)^{n}b)(ac - (ac)^{n+1})
= d(bd - (bd)^{n+1})^{l}(b - (bd)^{n}b)(ac - (ac)^{n+1}).
\]

Note that \( bd \in R^{nsD} \), i.e. \( bd - (bd)^{n+1} \in R^{nil} \). Hence, \( ac - (ac)^{n+1} \in R^{nil} \), i.e. \( ac \in R^{nsD} \). By Proposition 3.1 and the formula of [23, Theorem 2.1], we obtain \((ac)^{nsD} = d((bd)^{nsD})^{3}bac\).

In Theorem 3.7, let \( d = a \) and \( c = b \), then it is reduced as the following.

**Corollary 3.8** Let \( a, b \in R \) and \( n \in \mathbb{N} \). Then,

\[
ab \in R^{nsD} \iff ba \in R^{nsD}.
\]

In this case, \((ba)^{nsD} = b((ab)^{nsD})^{2}a\).

**Corollary 3.9** Let \( a, b \in R \) and \( n, k \in \mathbb{N} \). If \((ab)^{k} \in R^{nsD}\), then \((ba)^{k} \in R^{nsD}\).

**Proof** Since \( a((ba)^{k-1}b) = (ab)^{k} \in R^{nsD} \), by Corollary 3.8 we deduce that \((ba)^{k} \in R^{nsD}\).

Under the same hypotheses \( acd = dbd \) and \( dba = aca \), Jacobson’s lemma for the \( n \)-strong Drazin inverse is investigated as follows.

**Theorem 3.10** Let \( a, b, c, d \in R \) and \( n \in \mathbb{N} \). If \( acd = dbd \) and \( dba = aca \), then

\[
1 - ac \in R^{nsD} \iff 1 - bd \in R^{nsD}.
\]
Proof Suppose that $1 - ac \in R^{nsD}$. Then $(1 - ac) - (1 - ac)^{n+1} \in R^{nil}$. Now, by mathematical induction we prove that

$$(1 - bd) - (1 - bd)^{n+1})^{m+1} = -b((1 - ac) - (1 - ac)^{n+1})^m d(1 + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^{i-1}),$$

for any $m \in \mathbb{N}$.

Assume that $m = 1$. Since $acd = dbd$, we deduce that $(db)^i d = (ac)^i d$ for any $i \in \mathbb{N}$. Then we have

$$((1 - bd) - (1 - bd)^{n+1})^2 = (bd + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^i)(bd + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^i)$$

$$= (bd^2 + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^i) + (bd + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^i)$$

$$= b(ac + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (ac)^i d(1 + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^{i-1})$$

Assume that the conclusion holds for positive integer $m = l$. Now, we verify it for $m = l + 1$. One can see that

$$((1 - bd) - (1 - bd)^{n+1})^{l+2} = (1 - bd) - (1 - bd)^{n+1})^l ((1 - bd) - (1 - bd)^{n+1})$$

$$= b((1 - ac) - (1 - ac)^{n+1})^l d(1 + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^{i-1})(bd + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (bd)^i)$$

Note that $(1 - ac) - (1 - ac)^{n+1} \in R^{nil}$. Then $(1 - bd) - (1 - bd)^{n+1} \in R^{nil}$, i.e. $1 - bd \in R^{nsD}$.

Conversely, assume that $1 - bd \in R^{nsD}$. In order to prove $1 - ac \in R^{nsD}$, we will prove the following equality

$$((1 - ac) - (1 - ac)^{n+1})^{m+2} = d((1 - bd) - (1 - bd)^{n+1})^m bac(1 + \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (ac)^{i-1}),$$

for any $m \in \mathbb{N}$.

For the case $m = 1$. Note that $dba = aca$. Then we obtain $a(ca)^i = (db)^i a$, for any $i \in \mathbb{N}$. Hence, we
deduce that
\[
((1 - ac) - (1 - ac)^{n+1})^3 = -(ac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^i)(ac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^i)^2
\]
\[
= -((ac)^3 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i+2})(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -(ac)^2 c + \sum_{i=1}^{n+1} (-1)^i (n+1)_i a(ca)^{i+1}c(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -(db)^2ac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (db)^{i+1}ac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -d(db)bac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i d(bd)^{i+1}bac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= d((1 - db) - (1 - bd)^{n+1})bac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2.
\]
Assume that the conclusion holds for positive integer \(m = l\). Then, for the case \(m = l + 1\), we get
\[
((1 - ac) - (1 - ac)^{n+1})^{i+3} = -d((1 - bd) - (1 - bd)^{n+1})bac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -d((1 - bd) - (1 - bd)^{n+1})bac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -d((1 - bd) - (1 - bd)^{n+1})bacac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i b(ac)^i)(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -d((1 - bd) - (1 - bd)^{n+1})bacac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i b(ac)^i)(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -d((1 - bd) - (1 - bd)^{n+1})bacac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i b(ac)^i)(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= -d((1 - bd) - (1 - bd)^{n+1})bacac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (bd)^{i+1}bac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2
\]
\[
= d((1 - bd) - (1 - bd)^{n+1})bacac + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (bd)^{i+1}bac(1 + \sum_{i=1}^{n+1} (-1)^i (n+1)_i (ac)^{i-1})^2.
\]
Observe that \((1 - bd) - (1 - bd)^{n+1} \in R^{nil}\). Hence, \((1 - ac) - (1 - ac)^{n+1} \in R^{nil}\), as required.

**Corollary 3.11** Let \(a, b \in R\) and \(n \in \mathbb{N}\). Then,
\[
1 - ab \in R^{nsD} \iff 1 - ba \in R^{nsD}.
\]

4. The \(n\)-strong Drazin invertibility of the sum

Let \(p \in R\) be an idempotent \((p^2 = p)\). Then we can represent element \(a \in R\) as
\[
a = pap + pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p)
\]
or in the matrix form
\[
a = \begin{bmatrix}
a_1 & a_3 \\
a_4 & a_2
\end{bmatrix}_p,
\]
\[
8
\]
where $a_1 = pap$, $a_2 = (1 - p)a(1 - p)$, $a_3 = pa(1 - p)$ and $a_4 = (1 - p)ap$. For

$$x = \begin{bmatrix} x_1 & x_3 \\ x_4 & x_2 \end{bmatrix}_p \quad \text{and} \quad y = \begin{bmatrix} y_1 & y_3 \\ y_4 & y_2 \end{bmatrix}_p,$$

one can use usual matrix rules to obtain matrix forms of the sum $x + y$ and the product $xy$.

Remark that if $a_2 \in R^{nsD}$, then we have the following matrix representations relative to $p = aa^{nsD}$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \quad \text{and} \quad a^{nsD} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1^{nsD} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

where $a_1 \in (pRp)^{-1} \cap (pRp)^{nsD}$ and $a_2 \in ((1 - p)R(1 - p))^{nil}$.

In this section, our purpose is to investigate the $n$-strong Drazin invertibility of the sum of two elements in a ring. We start will a crucial auxiliary lemma.

**Lemma 4.1** Let $a, b \in R$ be such that $ab = 0$. Then,

$$a, b \in R^{nil} \iff a + b \in R^{nil}.$$

**Proof** Since $ab = 0$, we have

$$(a + b)^m = a^m + ba^{m-1} + b^2a^{m-2} + \cdots + b^m$$

for any $m \in \mathbb{N}$.

Suppose that $a, b \in R^{nil}$. Choose $k_1 \in \mathbb{N}$ satisfying $a^{k_1} = b^{k_1} = 0$. Then, we have $(a + b)^{2k_1} = 0$, which gives $a + b \in R^{nil}$.

On the contrary, assume that $(a + b)^{k_2} = 0$, for some $k_2 \in \mathbb{N}$. Then,

$$a^{k_2} + ba^{k_2-1} + b^2a^{k_2-2} + \cdots + b^{k_2} = 0.$$

Multiplying the preceding equality by $a$ from the left side (resp. by $b$ from the right side), we obtain $a^{k_2+1} = 0$ (resp. $b^{k_2+1} = 0$). Hence, we have $a, b \in R^{nil}$.

Now, we state the relationship between the $n$-strong Drazin invertibility of the elements $a, b$ and that of the sum $a + b$, under the condition $ab = 0$.

**Theorem 4.2** Let $n \in \mathbb{N}$ and $a, b \in R$ be such that $ab = 0$. Then,

$$a, b \in R^{nsD} \iff a + b \in R^{nsD}.$$

**Proof** By the hypothesis $ab = 0$, we have

$$x := (a + b) - (a + b)^{n+1} = (a - a^{n+1}) + (b - b^{n+1}) - (ba^n + b^2a^{n-1} + \cdots + b^n a) := x_1 + x_2 - x_3.$$

Note that $x_1(x_2 - x_3) = 0$, $x_3x_2 = 0$ and $x_3^2 = 0$. In view of Lemma 4.1, we get
Then, by Theorem 3.2 we obtain \( a, b \in R^{nsD} \) if and only if \( a + b \in R^{nsD} \).

**Remark 4.3** (1) For the Drazin invertibility, we have

\[
a, b \in R^D \iff a + b \in R^D,
\]

under the condition \( ab = ba = 0 \). In fact, the necessity can be seen from [9, Corollary 1]. Now, suppose that \( a + b \in R^D \). Then \( (a + b)^m = (a + b)^{m+1}R \cap R(a + b)^{m+1} \), for some \( m \in \mathbb{N} \). Hence, we have \( a^m + b^m = (a^{m+1} + b^{m+1})u = v(a^{m+1} + b^{m+1}) \), for some \( u, v \in R \). Multiplying the previous equality by \( a \) from the left side and right side respectively, we have \( a^{m+1} = a^{m+2}u = va^{m+2} \). Hence, \( a \in R^D \). Similarly, we can obtain \( b \in R^D \).

(2) By [3, Theorem 2.1], one can see that

\[
a, b \in R^D \implies a + b \in R^D,
\]

under the condition \( ab = 0 \). Now, we consider its converse. Assume that \( a + b \in R^D \). Then, we can obtain that \( a \) is right \( \pi \)-regular and \( b \) is left \( \pi \)-regular. Is \( a \) left \( \pi \)-regular? Is \( b \) right \( \pi \)-regular?

Next, we will consider the \( n \)-strong Drazin invertibility of the sum \( a + b \) under another new condition \( a^2b = aba \) and \( ab^2 = bab \), which is weaker than \( ab = ba \). Indeed, it is obvious that \( ab = ba \) imply \( a^2b = aba \) and \( ab^2 = bab \). However, the converse does not hold in general, which can be seen from the following example:

**Example 4.4** Let \( R = M_2(\mathbb{Z}_2) \), where \( \mathbb{Z}_2 \) denote the residue class ring modulo 2. Take \( a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \). Clearly, \( a^2b = aba \) and \( ab^2 = bab \). However, \( ab \neq ba \).

In order to prove our main result, we need the following lemmas.

**Lemma 4.5** Let \( a, b \in R^{nil} \). If \( a^2b = aba \) and \( ab^2 = bab \), then \( a + b \in R^{nil} \).

**Proof** By the hypothesis \( a, b \in R^{nil} \), there exists \( m \in \mathbb{N} \) such that \( a^m = 0 \) and \( b^m = 0 \). Since \( a^2b = aba \) and \( ab^2 = bab \), we can see that each of the monomials in the expansion of \( (a + b)^{3m} \) is either \( a^{k_1}b^{k_2}a^{k_3} \) or \( b^{k_1}a^{k_2}b^{k_3} \), where \( k_1 + k_2 + k_3 = l_1 + l_2 + l_3 = 3m \). Hence, \( (a + b)^{3m} = 0 \), which means \( a + b \in R^{nil} \).

**Lemma 4.6** Let \( x \in R \) and \( p^2 = p \in R \). If \( x \) has the representation \( x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \), then

\[
a \in (pRp)^{nil} \text{ and } b \in (1 - p)R(1 - p))^{nil} \iff x \in R^{nil}.
\]

**Proof** Assume that \( a \in (pRp)^{nil} \) and \( b \in ((1 - p)R(1 - p))^{nil} \). By a simple computation, we obtain \( x^k = \begin{bmatrix} a^k & f_k \\ 0 & b^k \end{bmatrix} \), for any \( k \in \mathbb{N} \), where \( f_k = \sum_{i=0}^{k-1} a^i cb^{k-1-i} \). Let \( a^{t_1} = 0 \) and \( b^{t_2} = 0 \), where \( t_1, t_2 \in \mathbb{N} \).

Then, we have \( x^{t_1 + t_2} = 0 \), i.e. \( x \in R^{nil} \). Conversely, it is clear.
Lemma 4.7 Let \( n \in \mathbb{N} \) and \( p^2 = p, x, y \in R \). If \( x \) and \( y \) have the representations
\[
x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p \quad \text{and} \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p},
\]
then
\[
a \in (pR_p)^{nsD} \quad \text{and} \quad b \in ((1 - p)R(1 - p))^{nsD} \iff x \in R^{nsD} \quad (\text{resp. } y \in R^{nsD}).
\]

Proof Observe that
\[
x - x^{n+1} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p - \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p^{n+1} = \begin{bmatrix} a - a^{n+1} & * \\ 0 & b - b^{n+1} \end{bmatrix}_p.
\]
By Lemma 4.6, it follows that
\[
a - a^{n+1} \in (pR_p)^{nil} \quad \text{and} \quad b - b^{n+1} \in ((1 - p)R(1 - p))^{nil} \iff x - x^{n+1} \in R^{nil}.
\]
Using Theorem 3.2, we complete the proof.

Lemma 4.8 Let \( n \in \mathbb{N} \) and \( a, b \in R^{nsD} \) be such that \( a^2b = aba \). Then \( ab \in R^{nsD} \).

Proof Since \( a^2b = aba \), by induction we can obtain \( (ab)^m = a^m b^m \) for any \( m \in \mathbb{N} \). Applying Theorem 3.2, we only need to prove
\[
ab - (ab)^{n+1} = (a - a^{n+1})b + a^{n+1}(b - b^{n+1}) := x + y \in R^{nil}.
\]
Note that
\[
yx = a^{n+1}(b - b^{n+1})(a - a^{n+1})b
= (a^n(ab) - (ab)^{n+1})(a - a^{n+1})b
= (a - a^{n+1})(a^n(ab) - (ab)^{n+1})b
= a^{n+1}(a - a^{n+1})b(a - a^{n+1})b
= (a - a^{n+1})ba^{n+1}(b - b^{n+1})
= xy.
\]
In addition, we can check that \( x^m = (a - a^{n+1})^m b^m \) and \( y^m = (a^{n+1})^m (b - b^{n+1})^m \) for any \( m \in \mathbb{N} \). Note that \( a - a^{n+1} \in R^{nil} \) and \( b - b^{n+1} \in R^{nil} \), which imply \( x \in R^{nil} \) and \( y \in R^{nil} \). Hence, \( x + y \in R^{nil} \) by Lemma 2.1(2).

Remark 4.9 In view of Lemma 4.8 and [26, Lemma 2], one can see that if \( n \in \mathbb{N} \) and \( a, b \in R^{nsD} \) be such that \( ab = ba \), then \( ab \in R^{nsD} \) and \( (ab)^{nsD} = b^{nsD}a^{nsD} \).

Now, we state our main result in this section as follows.

Theorem 4.10 Let \( n \in \mathbb{N} \) and \( a, b \in R^{nsD} \) be such that \( a^2b = aba \) and \( ab^2 = bab \). Then,
\[
1 + a^{nsD}b \in R^{nsD} \iff a + b \in R^{nsD}.
\]
Proof We consider the matrix representations of $a$ and $b$ relative to the idempotent $p = aa^{nsD}$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p,$$

where $a_1 \in (pRp)^{-1} \cap (pRp)^{nsD}$ and $a_2 \in ((1 - p)R(1 - p))^{nil}$.

The condition $a^2b = aba$ expressed in matrix form yields

$$\begin{bmatrix} a_1^2b_1 & a_1^2b_3 \\ a_2^2b_4 & a_2^2b_2 \end{bmatrix} = a^2b = aba = \begin{bmatrix} a_1b_1a_1 & a_1b_3a_2 \\ a_2b_4a_1 & a_2b_2a_2 \end{bmatrix}_p.$$

Thus, we have $a_1^2b_1 = a_1b_3a_2$, i.e. $b_3 = a_1^{-1}b_3a_2$, which implies $b_3 = a_1^{-m}b_3a_2^m$ for any $m \in \mathbb{N}$. Since $a_2 \in ((1 - p)R(1 - p))^{nil}$, we have $b_3 = 0$. Moreover, we can get $a_1b_1 = b_1a_1$ and $a_2^2b_2 = a_2b_2a_2$. Similarly, by $ab^2 = bab$ we obtain $a_2b_2^2 = b_2a_2b_2$. Therefore, we have

$$b = \begin{bmatrix} b_1 \\ b_4 \\ 0 \\ b_2 \end{bmatrix}_p \quad \text{and} \quad a + b = \begin{bmatrix} a_1 + b_1 \\ b_4 \\ a_2 + b_2 \end{bmatrix}_p.$$

Now, we prove that $a_2 + b_2 \in ((1 - p)R(1 - p))^{nsD}$. Since $b \in R^{nsD}$, by Lemma 4.7 we have $b_2 \in ((1 - p)R(1 - p))^{nsD}$. Let $p' = b_2a^{nsD}_2$. We consider the matrix representations of $b_2$ and $a_2$ relative to the idempotent $p'$:

$$b_2 = \begin{bmatrix} b_1' \\ b_4' \\ 0 \\ b_2' \end{bmatrix}_{p'} \quad \text{and} \quad a_2 = \begin{bmatrix} a_1' \\ a_4' \\ a_3' \\ a_2' \end{bmatrix}_{p'},$$

where $b_1' \in (p'Rp')^{-1} \cap (p'Rp')^{nsD}$ and $b_2' \in ((1 - p')R(1 - p'))^{nil}$. Note that $a_2^2b_2 = a_2b_2a_2$ and $a_2b_2^2 = b_2a_2b_2$. Then, we can obtain that $a_1' = 0$, $a_1'b_1' = b_1'a_1'$, $a_2'(b_2')^2 = b_2'a_2'b_2'$ and $(a_1')^2b_2' = a_1'b_2'a_2'$. Hence,

$$a_2 = \begin{bmatrix} a_1' \\ a_4' \\ a_3' \\ a_2' \end{bmatrix}_{p'} \quad \text{and} \quad a_2 + b_2 = \begin{bmatrix} a_1' + b_1' \\ 0 \\ a_2' + b_2' \end{bmatrix}_{p'}.$$

In order to show that $a_1' + b_1' = (p' + a_1'(b_1')^{-1})b_1' \in (p'Rp')^{nsD}$, by Lemma 4.8 we only need to prove $p' + a_1'(b_1')^{-1} \in (p'Rp')^{nsD}$. Since $a_2 \in ((1 - p)R(1 - p))^{nil}$, by Lemma 4.6 we obtain $a_1' \in (p'Rp')^{nil}$, which yields

$$(p' + a_1'(b_1')^{-1}) - (p' + a_1'(b_1')^{-1})^{n+1} = a_1'(b_1')^{-1}(p' - \sum_{i=1}^{n+1}(a_1')^{-1}(b_1')^{-1-i}) \in (p'Rp')^{nil}.$$

Hence, $p' + a_1'(b_1')^{-1} \in (p'Rp')^{nsD}$. Applying Lemma 4.5 to the nilpotent elements $a_2'$ and $b_2'$, we conclude that $a_2' + b_2' \in ((1 - p')R(1 - p'))^{nil}$, which implies $a_2' + b_2' \in ((1 - p')R(1 - p'))^{nsD}$. In view of Lemma 4.7, we obtain $a_2 + b_2 \in ((1 - p)R(1 - p))^{nsD}$. Then, by Lemma 4.7 again, it follows that $a + b \in R^{nsD}$ is equivalent to $a_1 + b_1 \in (pRp)^{nsD}$.

Note that

$$1 + a^{nsD}b = \begin{bmatrix} p + a_1^{-1}b_1 & 0 \\ 0 & 1 - p \end{bmatrix}_p.$$
Since \(1 - p \in ((1 - p)R(1 - p))^{nsD}\), then \(1 + a^{nsD}b \in R^{nsD}\) is equivalent to \(p + a_1^{-1}b_1 \in (pRp)^{nsD}\). Note that \(a_1 \in (pRp)^{nsD}\). Applying Corollary 3.5 we obtain \(a_1^{-1} = a_1^{nsD} \in (pRp)^{nsD}\). Hence, \(a_1 + b_1 = a_1(p + a_1^{-1}b_1) \in (pRp)^{nsD}\) is identical to \(p + a_1^{-1}b_1 \in (pRp)^{nsD}\) by Lemma 4.8. Hence, we conclude that \(a + b \in R^{nsD}\) if and only if \(1 + a^{nsD}b \in R^{nsD}\).

**Remark 4.11** In the proof of the necessity of Theorem 4.10, the condition \(ab^2 = bab\) was not used. However, if we drop it, then the sufficiency is not true in general, which will be shown in the next example:

**Example 4.12** Let \(n = 1 \in \mathbb{N}\) and \(R = M_3(\mathbb{C})\). Choose

\[
a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Then, we can check that \(a^2b = aba\), \(a^2 = 0\) and \(b^3 = 0\). Hence, \(a, b, 1 + a^{nsD}b = 1 \in R^{nsD}\). Note that the eigenvalues of \((a + b) - (a + b)^2\) are 0, \(\sqrt{3}i\) and \(-\sqrt{3}i\). Hence, \((a + b) - (a + b)^2 \notin R^{nil}\), which yields that \(a + b \notin R^{nsD}\). In addition, this example also illustrates that the condition \(ab^2 = bab\) of Lemma 4.5 cannot be dropped.

The following corollary can be directly derived from Theorem 4.10.

**Corollary 4.13** Let \(a, b \in R^{nsD}\) be such that \(ab = ba\). Then,

\[
a + b \in R^{nsD} \iff 1 + a^{nsD}b \in R^{nsD}.
\]

5. The \(n\)-strong Drazin invertibility of the matrix

In this section, as applications for our additive results of Section 4, we obtain some equivalent conditions for the \(n\)-strong Drazin invertibility of the matrix \(M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) over a ring. For the convenience of expressions, we assume that \(\sum_{i=j}^k s(i) = 0\) if \(k < j\), where \(s(i)\) is a function on \(i\), and \(a^0 = 1\) for \(a \in R\). For any nonnegative integer \(k\), by \(\lfloor k/2 \rfloor\) we denote the integer part of \(k/2\).

Firstly, we investigate the \(n\)-strong Drazin invertibility of some special antitriangular matrices over a ring.

**Proposition 5.1** Let \(n \in \mathbb{N}\) and \(M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) be such that \(ab = b, a = a^2\). Then,

1. \(bc \in R^{nil}\) if and only if \(M \in M_2(R)^{nsD}\).
2. If \(bc \in R^{nil}\), then \(M \in M_2(R)^{nsD}\).

**Proof** (1) Since \(ab = b\) and \(a = a^2\), we have

\[
M - M^2 = \begin{bmatrix} -bc & 0 \\ c - ca & -cb \end{bmatrix}.
\]
Hence, \( M - M^2 \in M_2(R)^{nil} \) is identical to \( bc \in R^{nil} \). From Theorem 3.2, it follows that \( bc \in R^{nil} \) if and only if \( M \in M_2(R)^{sD} \).

(2) This follows from item (1) and Corollary 3.6(1) directly.

**Remark 5.2** (1) In Proposition 5.1(1), if we change the condition “\( bc \in R^{nil} \)” to “\( bc \in R^{sD} \), then the conclusion does not hold in general. For example, take \( M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{C}) \). Obviously, 1 \( \in \mathbb{C}^{sD} \).

However, \( M \notin M_2(\mathbb{C})^{sD} \), since \( M - M^2 \notin M_2(\mathbb{C})^{nil} \).

(2) The converse of Proposition 5.1(2) is not true for \( n \geq 2 \) in general, which will be illustrated by the following example:

**Example 5.3** Let \( R = M_2(\mathbb{Z}_3) \). Choose \( a = b = 1 \in R \) and \( c = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \in R \). Then, it is easy to see \( bc \notin R^{nil} \). However, \( M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in M_2(R)^{2sD} \), since we can check that \((M - M^3)^2 = 0\).

**Theorem 5.4** Let \( n \in \mathbb{N} \) and \( a, b \in R \) be such that \( aba = 0 \). Then,

\[
M = \begin{bmatrix} a & a \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD} \iff a \in R^{nsD} \iff M' = \begin{bmatrix} a & b \\ a & 0 \end{bmatrix} \in M_2(R)^{nsD}.
\]

**Proof** We will only prove that \( a \in R^{nsD} \) is equivalent to \( M \in M_2(R)^{nsD} \), since the case for \( M' \) is similar.

Suppose that \( n = 1 \). Then, by the condition \( aba = 0 \) we have

\[
X := M - M^2 = \begin{bmatrix} a - a^2 - ab & a - a^2 \\ b - ba & -ba \end{bmatrix}
\]

and

\[
X^{m+1} = \begin{bmatrix} (a - a^2)^m(a - a^2 - ab + b) & (a - a^2)^m+1 \\ (b - ba)(a - a^2)^m - 1(a - a^2 - ab + b) & (b - ba)(a - a^2)^m \end{bmatrix}
\]

for any \( m \geq 2 \). Hence, \( X \in M_2(R)^{nil} \) if and only if \( a - a^2 \in R^{nil} \). Applying Theorem 3.2, we claim that \( a \in R^{sD} \) is equivalent to \( M \in M_2(R)^{sD} \).

The result for \( n \geq 2 \) follows analogously.

Next, we present an existence criterion for the \( n \)-strong Drazin inverse of the anti-triangular \( \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \), which will be used later.

**Theorem 5.5** Let \( n \in \mathbb{N} \) and \( M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R) \) be such that \( ab = 0 \). Then,

\[
a \in R^{nsD} \text{ and } N = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \in M_2(R)^{nil} \iff M \in M_2(R)^{nsD},
\]

where
\[ n_{11} = n_{22} = \frac{1-(-1)^n}{2}(b^{\frac{n+1}{2}})^2 + (1 - \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}})(b - \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}}), \]
\[ n_{12} = ((-1)^n - 1)b^{\frac{n+1}{2}}, \]
\[ n_{21} = ((-1)^n - 1)b^{\frac{n+1}{2}} + 1. \]

**Proof** Since \( ab = 0 \), then by induction we have

\[
X := M - M^{n+1} = \begin{bmatrix}
(a - a^{n+1}) - (t_1 + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}})(a - a^{n+1}) + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}}(t_1 + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}})
\end{bmatrix} \begin{bmatrix}
1 - a^{2}b^{\frac{n+1}{2}} + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}} - (t_3 + \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}})
\end{bmatrix}
\]

where

\[
t_1 = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} b^i a^{2i(\frac{n+1}{2}) - i + \frac{1-(-1)^n}{2}},
\]
\[
t_2 = \sum_{i=1}^{\frac{n+1}{2}} b^i a^{2i(\frac{n+1}{2}) - i + \frac{1-(-1)^n}{2}},
\]
\[
t_3 = \sum_{i=1}^{\frac{n+1}{2}} b^i a^{2i(\frac{n+1}{2}) - i + \frac{1-(-1)^n}{2}}.
\]

By a computation, we obtain

\[
X^2 = \begin{bmatrix}
\begin{array}{c c}
\frac{1}{2}a^{n+1} & \frac{1}{2}a^{n+1} \\
0 & 0
\end{array}
\end{bmatrix} + \begin{bmatrix}
\begin{array}{c c}
p_{12} & q_{11} \\
qu_{21} & qu_{22}
\end{array}
\end{bmatrix} + \begin{bmatrix}
\begin{array}{c c}
n_{11} & n_{12} \\
n_{21} & n_{22}
\end{array}
\end{bmatrix} := P + Q + N,
\]

where

\[
p_{12} = (a - a^{n+1})(1-a^{2\frac{n+1}{2}} + \frac{1-(-1)^n}{2}),
\]
\[
q_{11} = -(t_1 + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}})(a - a^{n+1}) + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}}t_1 - (1 - \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}})t_2,
\]
\[
q_{12} = -2t_1 + (t_1 + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}})a^{\frac{n+1}{2}} + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}}t_1 + \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}}t_3,
\]
\[
q_{21} = b(a - a^{n+1}) - bt_1 - (t_2 + \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}})(a - a^{n+1}) + \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}}t_1 + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}}t_2,
\]
\[
q_{22} = b(a - a^{n+1}) - bt_1 - (t_2 + \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}})(a - a^{n+1}) + \frac{1+(-1)^n}{2}b^{\frac{n+1}{2}}t_1 + \frac{1-(-1)^n}{2}b^{\frac{n+1}{2}}t_2,
\]
\[ q_{22} = -bt_3 - t_2 + (t_2 + \frac{1+(-1)^n}{2}b^{\frac{n+2}{2}} - b)a^2(\frac{1}{2} + \frac{1+(-1)^n}{2}b^{\frac{n+2}{2}}) + \frac{1+(-1)^n}{2}b^{\frac{n+2}{2}}t_3 + \frac{1+(-1)^n}{2}b^{\frac{n+2}{2}}t_1. \]

Note that \( P(Q + N) = 0, \) \( QN = 0 \) and \( Q^2 = 0. \) In view of Lemma 4.1, we have

\[ X \in M_2(R)^{nil} \iff X^2 \in M_2(R)^{nil} \iff P \in M_2(R)^{nil} \text{ and } Q + N \in M_2(R)^{nil} \iff P \in M_2(R)^{nil} \text{ and } N \in M_2(R)^{nil} \iff a - a^{n+1} \in R^{nil} \text{ and } N \in M_2(R)^{nil}. \]

In view of Theorem 3.2, we can conclude that \( a \in R^{nsD} \) and \( N \in M_2(R)^{nil} \) if and only if \( M \in M_2(R)^{nsD}. \)

Now, we state a special case of Theorem 5.5.

**Corollary 5.6** Let \( n = 2k \) \((k \in \mathbb{N})\) and let \( a, b \in R \) be such that \( ab = 0. \) Then,

\[ a \in R^{nsD} \text{ and } b \in R^{ksD} \iff M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD}. \]

**Proof** Let \( n = 2k \) in Theorem 5.5, we have \( N = \begin{bmatrix} b(1 - b^k)^2 & 0 \\ 0 & b(1 - b^k)^2 \end{bmatrix}. \) Then, one can see that

\[ N \in M_2(R)^{nil} \iff b(1 - b^k)^2 \in R^{nil} \iff b(1 - b^k)^2 \in R^{nil} \iff b - b^{k+1} \in R^{nil}. \]

Hence, we have that \( M \in M_2(R)^{nsD} \) is equivalent to \( a \in R^{nsD} \) and \( b \in R^{ksD}. \)

**Remark 5.7** By Corollary 5.6 and Corollary 3.6(3), we can see that

\[ M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD} \iff a \in R^{nsD} \text{ and } b \in R^{ksD}, \]

under the condition \( ab = 0 \) and \( n \) is one even number. However, the converse does not hold in general, which can be seen in the next example:

**Example 5.8** Let \( R = M_2(\mathbb{C}) \) and \( n = 2 \in \mathbb{N}. \) Setting \( M = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \in M_2(R), \) where \( b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in R. \) Obviously, \( b - b^3 = 0, \) which yields \( b \in R^{2sD}. \) However, we can check that \( M - M^3 \notin M_2(R)^{nil}, \) so we have \( M \notin M_2(R)^{2sD}. \)

Following the same strategy as in the proof of Theorem 5.5, we derive the equivalent condition for the \( n \)-strong Drazin invertibility of the transpose of the matrix \( M \) as follows:

**Theorem 5.9** Let \( n \in \mathbb{N} \) and \( M' = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \in M_2(R) \) be such that \( ab = 0. \) Then,

\[ a \in R^{nsD} \text{ and } N' = \begin{bmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{bmatrix} \in M_2(R)^{nil} \iff M' \in M_2(R)^{nsD}, \]

where \( n_{11}, n_{12}, n_{21}, \) and \( n_{22} \) are defined as in Theorem 5.5.
Combining Theorem 5.5 and Theorem 5.9, together with the equality $(N')^m = (N^m)'$ for any $m \in \mathbb{N}$, we obtain the relationship between the $n$-strong Drazin invertibility of the matrix $M$ and that of its transpose $M'$.

**Corollary 5.10** Let $n \in \mathbb{N}$ and let $a, b \in R$ be such that $ab = 0$. Then,

$$M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD} \iff M' = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \in M_2(R)^{nsD}.$$ 

In the rest of this section, applying the previous results we obtain some characterizations for the $n$-strong Drazin invertibility of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, under some conditions.

**Theorem 5.11** Let $n \in \mathbb{N}$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ be such that $abc = 0$ and $bd = 0$. Then,

$$a, d \in R^{nsD} \text{ and } T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \in M_2(R)^{nil} \iff M \in M_2(R)^{nsD},$$

where

$$t_{11} = t_{22} = \frac{1-(-1)^n}{2}((bc)^{\frac{n+1}{2}})^2 + (1 - \frac{1+(-1)^n}{2}(bc)^{\frac{1}{2}})(bc - \frac{1+(-1)^n}{2}(bc)^{\frac{n+2}{2}}),$$

$$t_{12} = ((-1)^n - 1)(bc)^{\frac{n+3}{2}} + 1,$$

$$t_{21} = ((-1)^n - 1)(bc)^{\frac{n+1}{2}}.$$ 

**Proof** We write $M = P + Q$, where

$$P = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$ 

The condition $bd = 0$ ensures $PQ = 0$. Note that $Q \in M_2(R)^{nsD}$ if and only if $d \in R^{nsD}$. In view of Theorem 4.2, we obtain $M \in M_2(R)^{nsD}$ is equivalent to $P \in M_2(R)^{nsD}$ and $d \in R^{nsD}$. 

Since

$$P = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & c \end{bmatrix} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix},$$

by Corollary 3.8 we have

$$P \in M_2(R)^{nsD} \iff P' := \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & bc \\ 1 & 0 \end{bmatrix} \in M_2(R)^{nsD}.$$ 

Since $abc = 0$, by Theorem 5.9 we obtain

$$a \in R^{nsD} \text{ and } T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \in M_2(R)^{nil} \iff P' \in M_2(R)^{nsD},$$

as required.

Now, we can derive some special cases of Theorem 5.11.
Corollary 5.12 Let \( n = 2k \) \((k \in \mathbb{N})\) and \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)\) be such that \( abc = 0 \) and \( bd = 0 \).

Then,

\[
a, d \in R^{nsD} \text{ and } bc \in R^{ksD} \iff M \in M_2(R)^{nsD}.
\]

Let \( k = 1 \) in Corollary 5.12, we have

Corollary 5.13 Let \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)\) be such that \( abc = 0 \) and \( bd = 0 \). Then,

\[
a, d \in R^H \text{ and } bc \in R^{sD} \iff M \in M_2(R)^H.
\]

Corollary 5.14 Let \( n \in \mathbb{N} \) and \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)\) be such that \( abc = 0 \), \( bd = 0 \) and \( bc = bca \).

Then,

\[
a, d \in R^{nsD} \iff M \in M_2(R)^{nsD}.
\]

Proof Using the condition \( abc = 0 \) and \( bc = bca \), we have \( T^2 = 0 \), where \( T \) is defined as in Theorem 5.11.

Remark 5.15 Let \( n \in \mathbb{N} \) and \( a, b, c, d \in R \) be such that \( abc = 0 \) and \( bd = 0 \). Then

\[
M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)^{nsD} \implies M' = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(R)^{nsD}
\]

does not hold in general, even if \( d = 0 \). For example:

Example 5.16 Let \( R = M_2(\mathbb{Z}_2) \) and \( n = 1 \in \mathbb{N} \). Choose \( M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in M_2(R)\), where

\[a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.
\]

Then, we can check that \((M - M^2)^2 = 0\) and \((M')^2 \notin M_2(R)^{nil}\). Hence, \( M \in M_2(R)^{sD}\). However, \( M' \notin M_2(R)^{sD}\).

Similar to the proof of Lemma 4.7 and using the representations of \([11, \text{Theorem 1}]\), we can obtain the following lemma.

Lemma 5.17 Let \( n \in \mathbb{N} \) and \( M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in M_2(R)\). Then

\[
a \in R^{nsD} \text{ and } d \in R^{nsD} \iff M \in M_2(R)^{nsD}.
\]

In this case,

\[
M^{nsD} = \begin{bmatrix} a^{nsD} & z \\ 0 & d^{nsD} \end{bmatrix},
\]

where

\[
z = \sum_{i=0}^{\text{ind}(d)-1} (a^{nsD})^{i+2}bd^i(1 - dd^{nsD}) + \sum_{i=0}^{\text{ind}(a)-1} (1 - aa^{nsD})a^ib(d^{nsD})^{i+2} - a^{nsD}bd^{nsD}.
\]
Applying Lemma 5.17, we may now state the following result.

**Theorem 5.18** Let $n \in \mathbb{N}$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ be such that $ca = 0$ and $cb = 0$. Then,

$$a, d \in R^{nsD} \iff M \in M_2(R)^{nsD}.$$  

**Proof** The matrix $M$ can be split as

$$M = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} := P + Q.$$  

The conditions $ca = 0$ and $cb = 0$ imply $PQ = 0$. Note that $P^2 = 0$. By Theorem 4.2 and Lemma 5.17, we conclude that $a, d \in R^{nsD}$ if and only if $M \in M_2(R)^{nsD}$.

The next theorem presents new conditions under which we give a characterization for the $n$-strong Drazin invertibility of the matrix $M$ over a ring.

**Theorem 5.19** Let $n \in \mathbb{N}$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ be such that $bc = cb = 0$ and $ca = dc$. Then,

$$a, d \in R^{nsD} \iff M \in M_2(R)^{nsD}.$$  

**Proof** Suppose that $a, d \in R^{nsD}$. Now, we consider the following splitting

$$M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} := P + Q.$$  

The conditions $bc = cb = 0$ and $ca = dc$ imply $PQ =QP$. In view of Lemma 5.17, we deduce that $P \in M_2(R)^{nsD}$ and

$$1 + P^{nsD}Q = \begin{bmatrix} 1 + zc \\ d^{nsD}c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $z$ is defined as in Lemma 5.17. Since $ca = dc$, by [10, Theorem 2] we obtain

$$ca^{nsD} = ca^D = d^Dc = d^{nsD}c.$$  

Hence, for any $m \in \mathbb{N}$, we have

$$b(d^{nsD})^mc = bc(a^{nsD})^m = 0,$$

and

$$bd^m(1 - dd^{nsD})c = bd^m c - bd^{m+1}d^{nsD}c = bca^m - bca^{m+1}a^{nsD} = 0.$$  

In addition, $a^{nsD}b^{nsD}c = a^{nsD}bca^{nsD} = 0$. Hence, $zc = 0$. Hence, $1 + P^{nsD}Q \in M_2(R)^{nsD}$. Applying Corollary 4.13, we deduce that $M \in M_2(R)^{nsD}$.

Conversely, suppose that $M \in M_2(R)^{nsD}$, i.e. $X := M - M^{n+1} \in M_2(R)^{nil}$. By induction, we can obtain

$$X = \begin{bmatrix} a - a^{n+1} & b - \sum_{i=0}^{n} a^{n-i}bd^i \\ c - (n+1)d^nc & d - d^{n+1} \end{bmatrix}.$$
By a computation, we have
\[ X^m = \left[ (a - a^{n+1})^m \\ (d - d^{n+1})^m \right], \]
for any \( m \in \mathbb{N} \). Therefore, we conclude that \( a - a^{n+1} \in R^{nil} \) and \( d - d^{n+1} \in R^{nil} \), which yield \( a, d \in R^{nsD} \), as required.

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