The relation between b-weakly compact operator and KB-operator

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Abstract: Our aim is to solve the problem asked by Bahramnezhad and Azar in "KB-operators on Banach lattices and their relationships with Dunford-Pettis and order weakly compact operators". We show that a continuous operator $R$ from a Banach lattice $N$ into a Banach space $M$ is a $b$-weakly compact operator if and only if $R$ is a $KB$-operator.

Key words: b-weakly compact operator, KB-operator

1. Introduction

In [6], Bahramnezhad and Azar defined a new classes of operators, named $KB$-operator and they have examined some of their properties and asked the following problem;

Problem 1.1 ([6], Problem 2.27) Give an operator $R$ from a Banach lattice $N$ into a Banach space $M$ which is a $KB$-operator but is not $b$-weakly compact.

We answer the question in the negative. A lot of properties and results on $b$-weakly compact operators were given in [2 − 5, 7]. Now, we recall the definitions of $b$-weakly compact operator and $KB$-operator.

Definition 1.2 Let $R$ be a continuous operator from a Banach lattice $N$ into a Banach space $M$.

(i) $R$ is called $KB$-operator if $R(x_n)$ has a norm convergent subsequence in $M$ for every positive increasing sequence $(x_n)$ of the closed unit ball $B_N$ of $N$.

(ii) $R$ is called $b$-weakly compact if $R(x_n)$ is norm convergent for every positive increasing sequence $(x_n)$ of the closed unit ball $B_N$ of $N$.

For the basic theory on vector lattices and for unexplained terminology we refer to [1, 8].

2. Section

We will prove that the classes of $KB$-operators and the $b$-weakly compact operators are the same.

Theorem 2.1 Let $R$ be an operator from a Banach lattice $N$ into a Banach space $M$. $R$ is a $b$-weakly compact operator if and only if $R$ is a $KB$-operator.

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well known canonical embedding of $N \in \mathbb{R}_+$ by Theorem 1.7 in [1].

On the other hand, if there exists a subsequence $(x_k)$ of $(x_n)$, let us define

$$\Psi : N'_+ \rightarrow \mathbb{R}_+, \ f \rightarrow \Psi(f) = \sup f(x_k).$$

For each $k$ and $f \in N'_+$ we have

$$f(x_k) = |f(x_k)| \leq \|f\| \|x_k\| \leq \|f\|.$$ 

Then, $\sup f(x_k) \in \mathbb{R}_+$. We claim that $\Psi$ is additive. To see this, let $f,g \in N'_+$.

$$\Psi(f + g) = \sup [(f + g)(x_k)] = \sup [f(x_k) + g(x_k)] \leq \sup f(x_k) + \sup g(x_k) = \Psi(f) + \Psi(g).$$

On the other hand, if $x_m, x_t \in (x_k)$, then pick $x_i \in (x_k)$ with $x_m \leq x_t$ and $x_t \leq x_i$, and note that

$$f(x_m) + g(x_t) \leq f(x_i) + g(x_i) \leq \sup [(f + g)(x_i)] = \Psi(f + g).$$

Using a well-known technique (e.g., [1, p.14]), we have $\Psi(f) + \Psi(g) \leq \Psi(f + g)$. Therefore, $\Psi$ is additive and by Theorem 1.7 in [1] $\Psi$ extends uniquely to a positive operator from $N'$ into $\mathbb{R}$ (we call $\Psi$ again). Hence, $\Psi \in N''$. It is easy to see that $\Psi$ is an upper bound of $(x_k)$ in $N''$ (where $(x_k)$ is the image of $(x_k)$ under the well known canonical embedding of $N$ into the bidual $N''$). There exists $G$ in $N''$ with $(x_k)'' \uparrow G$ as $N''$ is Dedekind complete. Since $(x_k)''(f) \rightarrow G(f)$ for each $f \in N'_+$, we have $(x_k)'' \rightarrow G$ with respect to $\sigma(N'', N')$. Thus, all subsequences of $(x_n)$ are convergent to the same limit $G$ with respect to $\sigma(N'', N')$. By the hypothesis, there exists a subsequence $(x_{n_k})$ of $(x_n)$ and $y \in M$ such that $R(x_{n_k}) \rightarrow y$ with respect to norm-topology.

This leads to $[R(x_{n_k})]'' \rightarrow y''$ in $M''$ with respect to norm-topology which implies $[R(x_{n_k})]'' \rightarrow y''$ in $M''$ with respect to $\sigma(M'', M')$. The continuity of $R'' : (N'', \sigma(N'', N')) \rightarrow (M'', \sigma(M'', M'))$ yields $R'' [(x_k)'' \rightarrow R''(G)$ and $[R(x_{n_k})]'' \rightarrow y''$ with respect to $\sigma(M'', M')$. Since $R'' [(x_{n_k})'' = [R(x_{n_k})]''$, we have $R''(G) = y''$. So, this means that every norm convergent a subsequence of $R(x_n)$ has the same norm limit. Now, we will show that $R(x_n) \rightarrow y$ in $M$ with respect to norm-topology. We assume that $R(x_n)$ does not converge to $y$. Thus, there exist $\varepsilon > 0$ and a subsequence $(x_m)$ of $(x_n)$ such that $\|R(x_m) - y\| > \varepsilon$ for all $m$. By the hypothesis and the above conclusion there exists a subsequence $(x_{m_k})$ of $(x_m)$ such that $R(x_{m_k}) \rightarrow y$ with respect to norm-topology, which is a contradiction.

\[ \square \]

References
